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Extended BRST cohomology, consistent deformations and anomalies of four-dimensional supersymmetric gauge theories

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Abstract: The local cohomology of an extended BRST differential which includes global $N = 1$ supersymmetry and Poincaré transformations is completely and explicitly computed in four-dimensional supersymmetric gauge theories with super-Yang-Mills multiplets, chiral matter multiplets and linear multiplets containing 2-form gauge potentials. In particular we determine to first order all $N = 1$ supersymmetric and Poincaré invariant consistent deformations of these theories that preserve the $N = 1$ supersymmetry algebra on-shell modulo gauge transformations, and all Poincaré invariant candidate gauge and supersymmetry anomalies. When the Yang-Mills gauge group is semisimple and no linear multiplets are present, we find that all such deformations can be constructed from standard superspace integrals and preserve the supersymmetry transformations in a formulation with auxiliary fields, and the candidate anomalies are exhausted by supersymmetric generalizations of the well-known chiral anomalies. In the general case there are additional deformations and candidate anomalies which are relevant especially to the deformation of free theories and the general classification of interaction terms in supersymmetric field theories.

Keywords: Gauge Symmetry, BRST Symmetry, Anomalies in Field and String Theories, Supersymmetric Effective Theories.

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Contents

1. Introduction

This work closes a gap in the analysis of four-dimensional globally supersymmetric gauge theories by deriving completely the local cohomology of an extended BRST differential which includes $N = 1$ supersymmetry and the Poincaré symmetries in addition to the standard ingredients related to the gauge symmetries and the field equations. We anal-

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yse theories with super-Yang-Mills multiplets for all compact gauge groups, chiral matter multiplets in linear representation of these groups, and linear multiplets containing 2-form gauge potentials. In particular this includes super-Yang-Mills theories with arbitrarily many abelian gauge fields, and free supersymmetric theories with any number of vector gauge fields and 2-form gauge potentials. The extended BRST differential involves thus constant ghosts for $N = 1$ supersymmetry and the Poincaré symmetries, ghost fields for the gauge symmetries, ghost-for-ghost fields for the reducible gauge symmetries of the 2-form gauge potentials, and antifields for the field equations, Noether identities and reducibility relations between the Noether operators.

Until now the cohomology under study was only examined for super-Yang-Mills theories (with chiral matter multiplets but without linear multiplets) in the restricted space of functionals with integrands of mass dimension four or smaller than four[1] and ghost numbers zero or one [1, 2]. The restrictions on the space of functionals were motivated by applications in the context of renormalization of power counting renormalizable theories. These applications are no longer the only arena of interest for BRST cohomological investigations: local BRST cohomology is now applied also in nonrenormalizable and effective field theories [3, 4], in the analysis of local conservation laws (characteristic cohomology) [5], and in particular in the study of consistent deformations of classical field theories [6].[2] Our analysis covers these applications because we shall compute the cohomology in the space of all local functionals, without restrictions on the dimension or ghost number.

The paper has been organized as follows. In section 2 we specify the theories under study and the extended BRST differential. In section 3 we define the cohomological problem and relate it to other useful cohomological groups. In section 4 we introduce suitable variables to compute the cohomology efficiently. In section 5 we discuss the extended BRST transformations of these variables and derive a related graded commutator algebra which is of crucial importance for the cohomology. The cohomology and the main steps of the computation are presented in section 6, the results most important for algebraic renormalization, anomalies and consistent deformations in section 7. In section 8 we comment on the cohomology in negative ghost numbers. In section 9 we show that our results do not depend on the formulation of supersymmetry used here, and in section 10 we discuss to which extend they depend on the lagrangian. The main text ends with a brief conclusion in section 11 and is supplemented by two appendices, the first of which contains conventions and notation (in particular a list of frequently used functions and operators can be found here), while the second outlines proofs of lemmas given in the main text.

2. Field content, lagrangian, extended BRST transformations

We denote the super Yang-Mills multiplets by $A_\mu^i, \lambda_\alpha^i$, the chiral multiplets by φ^s, χ_α^s, and the linear multiplets by $\phi^a, B_{\mu\nu}^a, \psi_\alpha^a$. The index i of the super Yang-Mills multiplets refers

[1]We refer here to the dimension assignments given in eq. (B.4).

[2]The classification of consistent deformations is particularly interesting for free theories because it yields the possible interaction terms that can be added to these theories in a manner consistent with the gauge symmetries, $N = 1$ supersymmetry and Poincaré invariance.

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to a basis of a reductive (= semisimple plus abelian) Lie algebra $\mathfrak{g}_{\rm YM}$ whose semisimple part (if any) is compact ($\mathfrak{g}_{\rm YM}$ is otherwise arbitrary; its semisimple or abelian part may vanish). We shall assume that this basis of $\mathfrak{g}_{\rm YM}$ has been chosen such that the Cartan-Killing metric on the semisimple part of $\mathfrak{g}_{\rm YM}$ is proportional to the unit matrix. The index s of the chiral multiplets refers to a representation of $\mathfrak{g}_{\rm YM}$ (which may be trivial, see below). The index a of the linear multiplets is not related to $\mathfrak{g}_{\rm YM}$. The Yang-Mills ghost fields are denoted by C^i, the ghost fields of the $B^a_{\mu\nu}$ by Q^a_μ and the corresponding ghost-for-ghost fields by R^a. Furthermore we introduce antifields for all these fields and constant ghosts c^μ, $c^{\mu\nu}$ and ξ^α for global spacetime translations, Lorentz transformations and supersymmetry transformations, respectively:

$$
\begin{aligned}
\text{fields:}\quad & \{\Phi^A\} = \{A^i_\mu, \lambda^i_\alpha, \bar\lambda^i_{\dot\alpha}, C^i, \phi^a, B^a_{\mu\nu}, \psi^a_\alpha, \bar\psi^a_{\dot\alpha}, Q^a_\mu, R^a, \varphi^s, \bar\varphi_s, \chi^s_\alpha, \bar\chi_{s\dot\alpha}\}\\
\text{antifields:}\quad & \{\Phi^*_A\} = \{A^{*\mu}_i, \lambda^{*\alpha}_i, \bar\lambda^{*\dot\alpha}_i, C^*_i, \phi^*_a, B^{*\mu\nu}_a, \psi^{*\alpha}_a, \bar\psi^{*\dot\alpha}_a, Q^{*\mu}_a, R^*_a, \varphi^*_s, \bar\varphi^{*s}, \chi^{*\alpha}_s, \bar\chi^{*s\dot\alpha}\}\\
\text{constant ghosts:}\quad & \{c^\mu, \xi^\alpha, \bar\xi^{\dot\alpha}, c^{\mu\nu}\}.
\end{aligned}
$$

λ^i_α, ψ^a_α and χ^s_α are the components of complex Weyl spinor fields, $\bar\lambda^i_{\dot\alpha}$, $\bar\psi^a_{\dot\alpha}$ and $\bar\chi_{s\dot\alpha}$ denote their complex conjugates. φ^s are complex scalar fields, $\bar\varphi_s$ their complex conjugates.[3] A^i_μ, ϕ^a, $B^a_{\mu\nu}$, C^i, Q^a_μ and R^a are real fields. The Poincaré ghosts c^μ and ${c_\mu}^\nu$ are real, the supersymmetry ghosts ξ^α are constant complex Weyl spinors, $\bar\xi^{\dot\alpha}$ is the complex conjugate of ξ^α. According to our conventions, the antifield $\bar\Phi^*$ of the complex conjugate of a field Φ is related to the complex conjugate $\overline{\Phi^*}$ of the antifield of Φ according to:

$$
\bar\Phi^* = -\overline{\Phi^*}\,. \tag{2.1}
$$

In particular the antifields of real fields are thus purely imaginary. $B^a_{\mu\nu}$, $B^{*\mu\nu}_a$ and $c^{\mu\nu}$ are antisymmetric in their spacetime indices:

$$
B^a_{\mu\nu} = -B^a_{\nu\mu}\,,\qquad B^{*\mu\nu}_a = -B^{*\nu\mu}_a\,,\qquad c^{\mu\nu} = -c^{\nu\mu}\,.
$$

The Grassmann parities $|\Phi^A|$ of the fields and constant ghosts are (one has $|\Phi^A| = |\bar\Phi^A|$):

$$
\begin{aligned}
&|A^i_\mu| = |\phi^a| = |B^a_{\mu\nu}| = |\varphi^s| = |R^a| = |\xi^\alpha| = 0,\\
&|\lambda^i_\alpha| = |\psi^a_\alpha| = |\chi^s_\alpha| = |C^i| = |Q^a_\mu| = |c^\mu| = |{c_\mu}^\nu| = 1\,.
\end{aligned}
$$

The ghost numbers of the fields and constant ghosts are

$$
\begin{aligned}
\mathrm{gh}(A^i_\mu) &= \mathrm{gh}(\lambda^i_\alpha) = \mathrm{gh}(B^a_{\mu\nu}) = \mathrm{gh}(\phi^a) = \mathrm{gh}(\psi^a_\alpha) = \mathrm{gh}(\varphi^s) = \mathrm{gh}(\chi^s_\alpha) = 0\\
\mathrm{gh}(C^i) &= \mathrm{gh}(Q^a_\mu) = \mathrm{gh}(c^\mu) = \mathrm{gh}({c_\mu}^\nu) = 1\\
\mathrm{gh}(R^a) &= 2\,.
\end{aligned}
$$

The Grassmann parity of an antifield is opposite to the Grassmann parity of the corresponding field, and the ghost numbers of a field and its antifield add up to -1,

$$
|\Phi^*_A| = |\Phi^A| + 1 \pmod 2\,,\qquad \mathrm{gh}(\Phi^*_A) = -1 - \mathrm{gh}(\Phi^A)\,.
$$

[3]For a complex field, the field and its complex conjugate are treated as independent variables (instead of the real and imaginary part).

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To avoid the writing of indices we shall occasionally use the notation φ, χ, ϕ, $B_{\mu\nu}$, ψ, $\bar{\psi}$, Q_μ, R, $\bar{\varphi}^*$, $\bar{\chi}^*$ for "column vectors" with entries $\varphi^s, \dots, \bar{\chi}^{*s}$. Analgously φ^*, χ^*, ϕ^*, $B^{*\mu\nu}$, ψ^*, $\bar{\psi}^*$, $Q^{*\mu}$, R^*, $\bar{\varphi}$, $\bar{\chi}$ denote "row vectors" with entries $\varphi^*_s, \dots, \bar{\chi}_s$. Transposition of such vectors is denoted by $(\ )^t$. The Lie algebra $\mathfrak{g}_{\mathrm{YM}}$ is represented on φ and χ by antihermitean matrices[4] T_i with real structure constants ${f_{ij}}^k$,

$$T_i^\dagger = -T_i, \quad [T_i, T_j] = {f_{ij}}^k T_k \ .$$

We shall compute the cohomology explicitly for the following simple lagrangians (the results for more general lagrangians are discussed in section 10):

$$\begin{aligned} L = &-\frac{1}{4}\delta_{ij}F^i_{\mu\nu}F^{j\mu\nu} + \frac{\mathrm{i}}{2}\delta_{ij}(\nabla_\mu\bar{\lambda}^i\bar{\sigma}^\mu\lambda^j - \bar{\lambda}^i\bar{\sigma}^\mu\nabla_\mu\lambda^j) + \\ &+ \frac{1}{2}\partial_\mu\phi^t\partial^\mu\phi - \frac{1}{2}H^t_\mu H^\mu + \frac{\mathrm{i}}{2}(\partial_\mu\bar{\psi}^t\bar{\sigma}^\mu\psi - \bar{\psi}^t\bar{\sigma}^\mu\partial_\mu\psi) + \\ &+ \nabla_\mu\bar{\varphi}\nabla^\mu\varphi + \frac{\mathrm{i}}{4}(\nabla_\mu\bar{\chi}\bar{\sigma}^\mu\chi - \bar{\chi}\bar{\sigma}^\mu\nabla_\mu\chi) + \\ &+ \frac{1}{2}\delta^{ij}(\bar{\varphi}T_i\varphi)(\bar{\varphi}T_j\varphi) + \bar{\varphi}T_i\chi\lambda^i - \bar{\lambda}^i\bar{\chi}T_i\varphi \end{aligned} \tag{2.2}$$

where the field strengths $F^i_{\mu\nu}$ and H^μ and the covariant derivatives ∇_μ are:

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + {f_{jk}}^i A^j_\mu A^k_\nu \tag{2.3}$$

$$H^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} \tag{2.4}$$

$$\nabla_\mu\lambda^i = \partial_\mu\lambda^i + {f_{jk}}^i A^j_\mu\lambda^k \,, \qquad \nabla_\mu\bar{\lambda}^i = \partial_\mu\bar{\lambda}^i + {f_{jk}}^i A^j_\mu\bar{\lambda}^k \tag{2.5}$$

$$\nabla_\mu\varphi = \partial_\mu\varphi + A^i_\mu T_i\varphi \,, \qquad \nabla_\mu\bar{\varphi} = \partial_\mu\bar{\varphi} - A^i_\mu\bar{\varphi}T_i \tag{2.6}$$

$$\nabla_\mu\chi = \partial_\mu\chi + A^i_\mu T_i\chi \,, \qquad \nabla_\mu\bar{\chi} = \partial_\mu\bar{\chi} - A^i_\mu\bar{\chi}T_i \,. \tag{2.7}$$

The extended BRST transformations of the fields and constant ghosts are:

$$s_{\mathrm{ext}}A^i_\mu = \partial_\mu C^i + {f_{jk}}^i A^j_\mu C^k + \hat{c}^\nu\partial_\nu A^i_\mu - {c_\mu}^\nu A^i_\nu - \mathrm{i}\xi\sigma_\mu\bar{\lambda}^i + \mathrm{i}\lambda^i\sigma_\mu\bar{\xi} \tag{2.8}$$

$$s_{\mathrm{ext}}\lambda^i_\alpha = -{f_{jk}}^i C^j\lambda^k_\alpha + \hat{c}^\mu\partial_\mu\lambda^i_\alpha - \frac{1}{2}c^{\mu\nu}(\sigma_{\mu\nu}\lambda^i)_\alpha - \delta^{ij}\xi_\alpha(\bar{\varphi}T_j\varphi + \lambda^*_j\xi + \bar{\xi}\bar{\lambda}^*_j) + (\sigma^{\mu\nu}\xi)_\alpha F^i_{\mu\nu} \tag{2.9}$$

$$s_{\mathrm{ext}}\phi = \hat{c}^\mu\partial_\mu\phi + \xi\psi + \bar{\psi}\bar{\xi} \tag{2.10}$$

$$s_{\mathrm{ext}}B_{\mu\nu} = \partial_\nu Q_\mu - \partial_\mu Q_\nu + \hat{c}^\rho\partial_\rho B_{\mu\nu} - {c_\mu}^\rho B_{\rho\nu} - {c_\nu}^\rho B_{\mu\rho} + 2\xi\sigma_{\mu\nu}\psi - 2\bar{\psi}\bar{\sigma}_{\mu\nu}\bar{\xi} \tag{2.11}$$

$$s_{\mathrm{ext}}\psi_\alpha = \hat{c}^\mu\partial_\mu\psi_\alpha - \frac{1}{2}c^{\mu\nu}(\sigma_{\mu\nu}\psi)_\alpha + (\sigma^\mu\bar{\xi})_\alpha(H_\mu - \mathrm{i}\partial_\mu\phi) \tag{2.12}$$

$$s_{\mathrm{ext}}\varphi = -C^iT_i\varphi + \hat{c}^\mu\partial_\mu\varphi + \xi\chi \tag{2.13}$$

$$s_{\mathrm{ext}}\chi_\alpha = -C^iT_i\chi_\alpha + \hat{c}^\mu\partial_\mu\chi_\alpha - \frac{1}{2}c^{\mu\nu}(\sigma_{\mu\nu}\chi)_\alpha - 4\xi_\alpha\bar{\xi}\bar{\chi}^* - 2\mathrm{i}(\sigma^\mu\bar{\xi})_\alpha\nabla_\mu\varphi \tag{2.14}$$

[4] Our analysis covers arbitrary antihermitean representations $\{T_i\}$, including trivial ones. Actually the use of antihermitean representations is not essential and only made to simplify the notation. All results hold analogously also for other representations.

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$$s_{\text{ext}} C^i = \frac{1}{2} f_{kj}{}^i C^j C^k + \hat{c}^\mu \partial_\mu C^i - 2\mathrm{i}\xi\sigma^\mu\bar{\xi} A^i_\mu \tag{2.15}$$

$$s_{\text{ext}} Q_\mu = \mathrm{i}\partial_\mu R + \hat{c}^\nu \partial_\nu Q_\mu - c_\mu{}^\nu Q_\nu - 2\mathrm{i}\xi\sigma^\nu\bar{\xi} B_{\mu\nu} + 2\mathrm{i}\xi\sigma_\mu\bar{\xi}\phi \tag{2.16}$$

$$s_{\text{ext}} R = \hat{c}^\mu \partial_\mu R - 2\xi\sigma^\mu\bar{\xi} Q_\mu \tag{2.17}$$

$$s_{\text{ext}} c^\mu = c_\nu{}^\mu c^\nu + 2\mathrm{i}\xi\sigma^\mu\bar{\xi} \tag{2.18}$$

$$s_{\text{ext}} c_\nu{}^\mu = -c_\nu{}^\rho c_\rho{}^\mu \tag{2.19}$$

$$s_{\text{ext}} \xi^\alpha = \frac{1}{2} c^{\mu\nu} (\xi\sigma_{\mu\nu})^\alpha \tag{2.20}$$

$$s_{\text{ext}} \bar{\xi}^{\dot{\alpha}} = -\frac{1}{2} c^{\mu\nu} (\bar{\sigma}_{\mu\nu}\bar{\xi})^{\dot{\alpha}} \tag{2.21}$$

where

$$\hat{c}^\mu = c^\mu - x^\nu c_\nu{}^\mu . \tag{2.22}$$

The transformations of the complex conjugate fields are obtained from those given above according to

$$s_{\text{ext}} \bar{\Phi} = (-)^{|\Phi|}\, \overline{s_{\text{ext}} \Phi} \, .$$

The extended BRST transformations of the antifields are obtained according to:

$$s_{\text{ext}} \Phi^*_A = \frac{\hat{\partial}^R L_{\text{ext}}}{\hat{\partial}\Phi^A} \tag{2.23}$$

$$L_{\text{ext}} = L - (s_{\text{ext}}\Phi^N)|_{\Phi^*=0}\Phi^*_N + \frac{1}{2}\delta^{ij}(\lambda^*_i\xi + \bar{\xi}\bar{\lambda}^*_i)(\lambda^*_j\xi + \bar{\xi}\bar{\lambda}^*_j) + 4\chi^*\xi\,\bar{\xi}\bar{\chi}^* \tag{2.24}$$

where $\hat{\partial}^R/\hat{\partial}\Phi^A$ is the Euler-Lagrange right derivative with respect to Φ^A. The extended BRST transformations of derivatives of the fields and antifields are obtained from those of the fields and antifields simply by prolongation, i.e., by using

$$[s_{\text{ext}}, \partial_\mu] = 0 \, .$$

By construction s_{ext} squares to zero on all fields, antifields and constant ghosts,

$$s^2_{\text{ext}} = 0 \, .$$

Comment. The terms in (2.15), (2.16) and (2.18) which are bilinear in the supersymmetry ghosts reflect that the commutators of supersymmetry transformations contain gauge transformations and spacetime translations. In addition these commutators contain terms which vanish only on-shell. This is reflected by the antifield dependent terms in the extended BRST transformations (2.9) and (2.14). Schematically one has

$$[\text{susy transformation}, \text{susy transformation}] \approx \text{translation} + \text{gauge transformation}$$

where $\approx$ is equality on-shell, defined according to

$$X \approx Y \quad :\Leftrightarrow \quad X - Y = \sum_k Z^{\mu_1 \ldots \mu_k A} \partial_{\mu_1} \ldots \partial_{\mu_k} \frac{\hat{\partial}^R L}{\hat{\partial}\Phi^A} \, .$$

(Here X, Y and $Z^{\mu_1\dots\mu_k A}$ may depend on the fields and their derivatives; $Z^{\mu_1\dots\mu_k A}\times \partial_{\mu_1}\dots\partial_{\mu_k}\,\hat\partial^R L/\hat\partial\Phi^A$ vanishes on-shell, i.e., for all solutions of the fields equations, because L does not depend on ghosts). The Poincaré, supersymmetry and gauge transformations form thus an "open algebra" according to standard terminology. By introducing additional fields one may "close" and simplify the algebra, but this is irrelevant to the cohomology, see section 9.

3. Cohomological problem and descent equations

The primary goal is the determination of the local cohomological groups $H^{g,4}(s_{\mathrm{ext}}|d)$, i.e., the cohomology of s_{ext} modulo the exterior derivative $d = dx^\mu\partial_\mu$ in the space of local 4-forms with ghost numbers g. Unless differently specified, the term "local p-forms" is in this paper reserved for exterior p-forms $dx^{\mu_1}\dots dx^{\mu_p}\omega_{\mu_1\dots\mu_p}$ (the differentials are treated as Grassmann odd quantities) where $\omega_{\mu_1\dots\mu_p}$ can depend on the fields, antifields, their derivatives, constant ghosts and explicitly on the spacetime coordinates such that the overall number of derivatives of fields and antifields is finite, without further restriction on this number or on the order of derivatives that may occur.[5] The precise mathematical setting of the cohomological problem is made in the jet spaces associated to the fields and antifields, see [7] (the constant ghosts are just added as local coordinates of these jet spaces). The cocycles of $H^{g,4}(s_{\mathrm{ext}}|d)$ are local 4-forms with ghost number g, denoted by $\omega^{g,4}$, which are s_{ext}-closed up to d-exact forms $d\omega^{g+1,3}$ where $\omega^{g+1,3}$ is a local 3-form with ghost number $g+1$,

$$s_{\mathrm{ext}}\omega^{g,4} + d\omega^{g+1,3} = 0\,. \tag{3.1}$$

Coboundaries of $H^{g,4}(s_{\mathrm{ext}}|d)$ are local 4-forms $\omega^{g,4} = s_{\mathrm{ext}}\omega^{g-1,4} + d\omega^{g,3}$, where $\omega^{g-1,4}$ and $\omega^{g,3}$ are local forms with form-degree and ghost number indicated by their superscripts.

As usual, (3.1) implies descent equations for s_{ext} and d which relate $H^{*,4}(s_{\mathrm{ext}}|d)$ to $H(s_{\mathrm{ext}})$ and $H(s_{\mathrm{ext}}+d)$, the cohomologies of s_{ext} and $(s_{\mathrm{ext}}+d)$ in the space of local forms (see section 9 of [7] for a general discussion). In the present case these relations are very direct. To describe them precisely, we define the space $\mathfrak{F}$ of local functions (0-forms) that depend on the c^μ and x^μ only via the combinations $\hat c^\mu = c^\mu - x^\nu c_\nu{}^\mu$, and the space $\mathfrak{C}$ of polynomials in the constant ghosts,

$$\mathfrak{F} := \{f(\hat c^\mu,\xi^\alpha,\bar\xi^{\dot\alpha},c^{\mu\nu},\Phi^A,\Phi^*_A,\partial_\mu\Phi^A,\partial_\mu\Phi^*_A,\partial_\mu\partial_\nu\Phi^A,\partial_\mu\partial_\nu\Phi^*_A,\dots)\} \tag{3.2}$$
$$\mathfrak{C} := \{f(c^\mu,\xi^\alpha,\bar\xi^{\dot\alpha},c^{\mu\nu})\}\,. \tag{3.3}$$

$\mathfrak{F}$ is mapped by s_{ext} to itself ($s_{\mathrm{ext}}\mathfrak{F}\subseteq\mathfrak{F}$), because the s_{ext}-transformations of all fields, antifields, their derivatives and of the variables $\hat c^\mu,\xi^\alpha,\bar\xi^{\dot\alpha},c^{\mu\nu}$ are contained in $\mathfrak{F}$: s_{ext} acts on all these variables according to $s_{\mathrm{ext}} = \hat c^\mu\partial_\mu + \cdots$ where the nonwritten terms do not involve c^μ or x^μ. In particular, this holds for $s_{\mathrm{ext}}\hat c^\mu$:

$$s_{\mathrm{ext}}\hat c^\mu = \hat c^\nu\partial_\nu\hat c^\mu + 2\mathrm{i}\xi\sigma^\mu\bar\xi = -\hat c^\nu c_\nu{}^\mu + 2\mathrm{i}\xi\sigma^\mu\bar\xi\,. \tag{3.4}$$

[5] When dealing with a more complicated lagrangian than (2.2) (especially with an effective lagrangian), one may have to adapt the definition of local forms to the lagrangian, see section 10.

Since $\mathfrak{E}$ is also mapped by $s_{\rm ext}$ to itself ($s_{\rm ext}\mathfrak{E} \subseteq \mathfrak{E}$), both the cohomology $H(s_{\rm ext}, \mathfrak{F})$ of $s_{\rm ext}$ in $\mathfrak{F}$ and the cohomology $H(s_{\rm ext}, \mathfrak{E})$ of $s_{\rm ext}$ in $\mathfrak{E}$ are well-defined. The relation between $H^{*,4}(s_{\rm ext}|d)$, $H(s_{\rm ext} + d)$ and $H(s_{\rm ext}, \mathfrak{F})$ can now be described as follows:

Lemma 1. *$H(s_{\rm ext}, \mathfrak{F})$, $H(s_{\rm ext} + d)$ and $H^{*,4}(s_{\rm ext}|d) \oplus H(s_{\rm ext}, \mathfrak{E})$ are isomorphic:*

$$H^g(s_{\rm ext}, \mathfrak{F}) \simeq H^g(s_{\rm ext} + d) \simeq H^{g-4,4}(s_{\rm ext}|d) \oplus H^g(s_{\rm ext}, \mathfrak{E}) \,, \tag{3.5}$$

where the degree g in $H^g(s_{\rm ext}, \mathfrak{F})$, $H^g(s_{\rm ext}, \mathfrak{E})$ and $H^{g,4}(s_{\rm ext}|d)$ is the ghost number, while in $H^g(s_{\rm ext} + d)$ it is the sum of the ghost number and the form-degree. The representatives of $H^g(s_{\rm ext} + d)$ can be obtained from those of $H^g(s_{\rm ext}, \mathfrak{F})$ by substituting $c^\mu + dx^\mu$ for c^μ, the representatives of $H^{g,4}(s_{\rm ext}|d)$ are the 4-forms contained in the representatives of $H^{g+4}(s_{\rm ext} + d)/H^{g+4}(s_{\rm ext}, \mathfrak{E})$.

Comments.

(*a*) We shall compute $H(s_{\rm ext}, \mathfrak{F})$ and derive $H^{*,4}(s_{\rm ext}|d)$ from it according to the lemma. The representatives of $H^{g+4}(s_{\rm ext} + d)$ are the solutions of the descent equations (see proof of the lemma).

(*b*) The relations between $H^{*,4}(s_{\rm ext}|d)$, $H(s_{\rm ext} + d)$ and $H(s_{\rm ext}, \mathfrak{F})$ are much simpler than the relations between $H^{*,4}(s|d)$, $H(s + d)$ and $H(s)$ where s is the standard (non-extended) BRST differential for the theories under study. The reason is that $s_{\rm ext}$ and $s_{\rm ext} + d$ are directly related because $s_{\rm ext}$ contains the spacetime translations ($s_{\rm ext} + d$ arises on all fields and antifields from $s_{\rm ext}$ by substituting $c^\mu + dx^\mu$ for c^μ). As a consequence $H(s_{\rm ext}, \mathfrak{F})$ contains already the complete structure of the descent equations for $s_{\rm ext}$ and d. In contrast, s and $s + d$ are truly different and the descent equations for s and d are only contained in $H(s+d)$ but not in $H(s)$. Hence, $H(s_{\rm ext}, \mathfrak{F})$ is more similar to $H(s + d)$ than to $H(s)$. In particular, $\hat{c}^\mu$ plays in $H(s_{\rm ext}, \mathfrak{F})$ a role similar to dx^μ in $H(s+d)$ (apart from the fact that $s_{\rm ext}\hat{c}^\mu$ does not vanish, in contrast to $(s + d)dx^\mu$). This role of $\hat{c}^\mu$ is also similar to the role of the diffeomorphism ghosts in gravitational theories, see [8, 9].

(*c*) As a side remark, which is related to the previous comment, I note that $H(s_{\rm ext})$ (the cohomology of $s_{\rm ext}$ in the space of all local forms rather than only in $\mathfrak{F}$) and $H^{*,p}(s_{\rm ext}|d)$ for all $p > 0$ (rather than only for $p = 4$) can also be directly derived from $H(s_{\rm ext}, \mathfrak{F})$. The nontrivial representatives of $H(s_{\rm ext})$ are linear combinations of the nontrivial representatives of $H(s_{\rm ext}, \mathfrak{F})$ with coefficients that are ordinary differential forms $\omega(dx, x)$ on $\mathbb{R}^4$ (independent of fields, antifields or constant ghosts) which can be assumed not to be d-exact. The nontrivial representatives of $H^{*,p}(s_{\rm ext}|d)$ for $p > 0$ are linear combinations of nontrivial representatives of $H(s_{\rm ext})$ with form-degree p ("solutions with a trivial descent") and of the p-form part contained in nontrivial representatives of $H(s_{\rm ext} + d)/H(s_{\rm ext}, \mathfrak{E})$ ("solutions with a non-trivial descent"). This can be proved as analogous results in Einstein-Yang-Mills theory, see section 6 of [9] (the role of the space $\mathcal{A}$ in [9] is now taken by $\mathfrak{F}$, the role of the diffeomorphism ghosts by the $\hat{c}^\mu$). Note that for $p = 4$ the statement on $H^{*,p}(s_{\rm ext}|d)$ is in agreement with lemma 1 because 4-forms $\omega(dx, x)$ are d-exact in $\mathbb{R}^4$.

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(*d*) A gauge fixing need not be specified because it does not affect the cohomology (see, e.g., sections 2.6 and 2.7 of [7]).

4. Change of variables

To compute $H(s_{\rm ext},\mathfrak{F})$ we shall follow a strategy [10, 11] based on new jet coordinates u^ℓ, v^ℓ and w^I which satisfy

$$s_{\rm ext}u^\ell=v^\ell\,,\qquad s_{\rm ext}w^I=r^I(w)\,. \tag{4.1}$$

The important requirement here is that $r^I(w)$ is a function of the w's only. To construct such jet coordinates we use operations ∂_+^+, ∂_-^+, ∂_+^-, ∂_-^- which are defined as follows [12]: let Z_n^m be a Lorentz-irreducible multiplet of fields or antifields with m undotted and n dotted spinor indices,

$$Z_n^m\equiv\{Z_{\alpha_1\cdots\alpha_n}^{\dot\alpha_1\cdots\dot\alpha_m}\}\,,\qquad Z_{\alpha_1\cdots\alpha_n}^{\dot\alpha_1\cdots\dot\alpha_m}=Z_{(\alpha_1\cdots\alpha_n)}^{(\dot\alpha_1\cdots\dot\alpha_m)}\,;$$

we define ∂_+^+, ∂_-^+, ∂_+^-, ∂_-^- and $\Box$ according to

$$\begin{aligned}
\partial_+^+Z_n^m &\equiv\left\{\partial_{(\alpha_0}^{(\dot\alpha_0}Z_{\alpha_1\cdots\alpha_n)}^{\dot\alpha_1\cdots\dot\alpha_m)}\right\}\\
\partial_+^-Z_n^m &\equiv\left\{m\partial_{\dot\alpha_m(\alpha_0}Z_{\alpha_1\cdots\alpha_n)}^{\dot\alpha_1\cdots\dot\alpha_m}\right\}\\
\partial_-^+Z_n^m &\equiv\left\{n\partial^{\alpha_n(\dot\alpha_0}Z_{\alpha_1\cdots\alpha_n}^{\dot\alpha_1\cdots\dot\alpha_m)}\right\}\\
\partial_-^-Z_n^m &\equiv\left\{mn\partial_{\dot\alpha_m}^{\alpha_n}Z_{\alpha_1\cdots\alpha_n}^{\dot\alpha_1\cdots\dot\alpha_m}\right\}\\
\Box Z_n^m &\equiv\left\{\partial_\mu\partial^\mu Z_{\alpha_1\cdots\alpha_n}^{\dot\alpha_1\cdots\dot\alpha_m}\right\}=\left\{\frac{1}{2}\partial_{\beta\dot\beta}\partial^{\dot\beta\beta}Z_{\alpha_1\cdots\alpha_n}^{\dot\alpha_1\cdots\dot\alpha_m}\right\}.
\end{aligned} \tag{4.2}$$

Using these operations and the notation

$$A\equiv\{A^{i\dot\alpha}{}_\alpha\}\,,\ B^{(+)}\equiv\{\sigma^{\mu\nu}_{\alpha\beta}B^a_{\mu\nu}\}\,,\ B^{(-)}\equiv\{\bar\sigma^{\mu\nu\dot\alpha\dot\beta}B^a_{\mu\nu}\}\,,\ H\equiv\{H^{a\dot\alpha}{}_\alpha\}\,,\ Q\equiv\{Q^{a\dot\alpha}{}_\alpha\}\ \text{etc.}\,,$$

we define the following jet variables $u^\ell,v^\ell,w^I_{(0)}$:

$$\begin{aligned}
\{u^\ell\}=\bigcup_{p=0}^\infty\bigcup_{q=0}^\infty\Big(&\Box^p(\partial_+^+)^qA\cup\Box^p\partial_-^-A\\
&\cup\Box^p(\partial_+^+)^qC^*\\
&\cup\Box^p(\partial_+^+)^qA^*\cup\Box^p(\partial_+^+)^q\partial_-^+A^*\cup\Box^p(\partial_+^+)^q\partial_+^-A^*\\
&\cup\Box^p(\partial_+^+)^q\lambda^*\cup\Box^p(\partial_+^+)^q\partial_-^+\lambda^*\cup\Box^p(\partial_+^+)^q\bar\lambda^*\cup\Box^p(\partial_+^+)^q\partial_+^-\bar\lambda^*\\
&\cup\Box^p(\partial_+^+)^qB^{(+)}\cup\Box^p(\partial_+^+)^qB^{(-)}\cup\Box^p(\partial_+^+)^q\partial_+^-B^{(-)}\\
&\cup\Box^p(\partial_+^+)^qQ\cup\Box^p\partial_-^-Q\\
&\cup\Box^p(\partial_+^+)^qR^*\\
&\cup\Box^p(\partial_+^+)^qQ^*\cup\Box^p(\partial_+^+)^q\partial_-^+Q^*\cup\Box^p(\partial_+^+)^q\partial_+^-Q^*\\
&\cup\Box^p(\partial_+^+)^q\partial_+^-B^{*(-)}\\
&\cup\Box^p(\partial_+^+)^q\phi^*\\
&\cup\Box^p(\partial_+^+)^q\psi^*\cup\Box^p(\partial_+^+)^q\partial_-^+\psi^*\cup\Box^p(\partial_+^+)^q\bar\psi^*\cup\Box^p(\partial_+^+)^q\partial_+^-\bar\psi^*
\end{aligned}$$

$$
\begin{aligned}
&\cup \Box^p(\partial_+^+)^q\varphi^* \cup \Box^p(\partial_+^+)^q\bar\varphi^* \\
&\cup \Box^p(\partial_+^+)^q\chi^* \cup \Box^p(\partial_+^+)^q\partial_-^+\chi^* \cup \Box^p(\partial_+^+)^q\bar\chi^* \cup \Box^p(\partial_+^+)^q\partial_+^-\bar\chi^*\Big) \qquad (4.3)
\end{aligned}
$$

$$
\{v^\ell\} = \{s_{\mathrm{ext}}u^\ell\} \qquad (4.4)
$$

$$
\begin{aligned}
\{w^I_{(0)}\} =& \{C, R, \hat c^\mu, c^{\mu\nu}\ (\mu<\nu), \xi^\alpha, \bar\xi^{\dot\alpha}\} \\
&\cup \bigcup_{q=0}^{\infty}\Big((\partial_+^+)^q\partial_+^- A \cup (\partial_+^+)^q\partial_-^+ A \cup (\partial_+^+)^q\lambda \cup (\partial_+^+)^q\bar\lambda \\
&\quad \cup (\partial_+^+)^q H \cup (\partial_+^+)^q\phi \cup (\partial_+^+)^q\psi \cup (\partial_+^+)^q\bar\psi \\
&\quad \cup (\partial_+^+)^q\varphi \cup (\partial_+^+)^q\bar\varphi \cup (\partial_+^+)^q\chi \cup (\partial_+^+)^q\bar\chi\Big). \qquad (4.5)
\end{aligned}
$$

Lemma 2. *The u's, $w_{(0)}$'s and the linearized v's form a basis of the vector space (over $\mathbb{C}$) spanned by the fields, antifields, all their independent derivatives and the $\hat c^\mu, c^{\mu\nu}, \xi^\alpha, \bar\xi^{\dot\alpha}$.*

This implies that the u's, v's and $w_{(0)}$'s can be used as new jet coordinates substituting for the fields, antifields, their derivatives and constant ghosts. They do not have the desired quality (4.1) but can be extended to variables with this quality by means of an algorithm given in [11]:

Lemma 3. *The algorithm described in section 2 of [11] completes the $w^I_{(0)}$ to local functions w^I such that (4.1) holds.*

Eqs. (4.1) will allow us to compute the cohomology solely in terms of the w's (see section 6.2). We introduce the following notation for them:

$$
\begin{aligned}
\{w^I\} =& \{\hat C, \hat R, \hat c^\mu, c^{\mu\nu}\ (\mu<\nu), \xi^\alpha, \bar\xi^{\dot\alpha}\} \cup \{\hat T^\tau\} \\
\{\hat T^\tau\} =& \bigcup_{q=0}^{\infty}\Big((\hat\nabla_+^+)^q\hat F^{(+)} \cup (\hat\nabla_+^+)^q\hat F^{(-)} \cup (\hat\nabla_+^+)^q\hat\lambda \cup (\hat\nabla_+^+)^q\hat{\bar\lambda} \\
&\cup (\hat\nabla_+^+)^q\hat H \cup (\hat\nabla_+^+)^q\phi \cup (\hat\nabla_+^+)^q\hat\psi \cup (\hat\nabla_+^+)^q\hat{\bar\psi} \\
&\cup (\hat\nabla_+^+)^q\varphi \cup (\hat\nabla_+^+)^q\bar\varphi \cup (\hat\nabla_+^+)^q\hat\chi \cup (\hat\nabla_+^+)^q\hat{\bar\chi}\Big). \qquad (4.6)
\end{aligned}
$$

with an obvious correspondence to the variables in (4.5) [$\hat F^{(\pm)}$ corresponds to $\partial_\pm^\mp A$]. The $\hat T$'s may be called generalized tensor fields because their antifield independent parts are ordinary gauge covariant tensor fields. In addition they contain terms which depend on antifields such that $s_{\mathrm{ext}}w^I$ contains no terms that vanish on-shell. The $\hat\nabla_\mu$ can be viewed as generalizations of the ordinary covariant derivatives ∇_μ and are related to the latter as follows: the antifield independent part of $\hat\nabla_\mu f(\hat T)$ coincides on-shell with ∇_μ acting on the antifield independent part of $f(\hat T)$:

$$
\Big[\hat\nabla_\mu f(\hat T)\Big]_{\Phi^*=0} \approx \nabla_\mu\Big[f(\hat T)\Big]_{\Phi^*=0}. \qquad (4.7)
$$

For instance, the antifield independent part of $\hat\nabla_\mu\hat\nabla_\nu\phi$ is $(\partial_\mu\partial_\nu - \frac{1}{4}\eta_{\mu\nu}\Box)\phi$. The explicit form of the $\hat T$'s in terms of the original variables (fields, antifields, their derivatives and the constant ghosts) is somewhat involved. Fortunately we need not compute them explicitly to perform the cohomological analysis because their existence is guaranteed by lemma 3, and their extended BRST transformations can be directly obtained from the transformations of the $w_{(0)}$'s as we shall see in the following section. Nevertheless, for later purpose and to illustrate the structure of the new variables, we list a few w's explicitly:

$$\hat{C}^i = C^i + A^i_\mu \hat{c}^\mu \tag{4.8}$$

$$\hat{R} = R + \mathrm{i}Q_\mu \hat{c}^\mu + \frac{\mathrm{i}}{2} B_{\mu\nu}\hat{c}^\nu\hat{c}^\mu \tag{4.9}$$

$$\hat{\lambda}^i_\alpha = \lambda^i_\alpha - \frac{\mathrm{i}}{4}\delta^{ij}\hat{c}^\mu(\sigma_\mu\bar{\lambda}^*_j)_\alpha \tag{4.10}$$

$$\hat{F}^{i(+)}_{\alpha\beta} = \sigma^{\mu\nu}_{\alpha\beta}F^i_{\mu\nu} - 2\delta^{ij}\xi_{(\alpha}\lambda^*_{\beta)j} + \frac{2}{3}\delta^{ij}\hat{c}^\mu\sigma_{\mu\nu\alpha\beta}A^{*\nu}_j - \frac{1}{6}\delta^{ij}\hat{c}^\mu\hat{c}^\nu\sigma_{\mu\nu\alpha\beta}C^*_j \tag{4.11}$$

$$\hat{F}^{i(-)}_{\dot{\alpha}\dot{\beta}} = \bar{\sigma}^{\mu\nu}_{\dot{\alpha}\dot{\beta}}F^i_{\mu\nu} - 2\delta^{ij}\bar{\xi}_{(\dot{\alpha}}\bar{\lambda}^*_{\dot{\beta})j} + \frac{2}{3}\delta^{ij}\hat{c}^\mu\bar{\sigma}_{\mu\nu\dot{\alpha}\dot{\beta}}A^{*\nu}_j - \frac{1}{6}\delta^{ij}\hat{c}^\mu\hat{c}^\nu\bar{\sigma}_{\mu\nu\dot{\alpha}\dot{\beta}}C^*_j \tag{4.12}$$

$$\hat{\psi}^a_\alpha = \psi^a_\alpha - \frac{\mathrm{i}}{4}\delta^{ab}\hat{c}^\mu(\sigma_\mu\bar{\psi}^*_b)_\alpha \tag{4.13}$$

$$\hat{H}^a_\mu = H^a_\mu + \frac{3}{4}\delta^{ab}(\psi^*_b\sigma_\mu\bar{\xi} - \xi\sigma_\mu\bar{\psi}^*_b) - \varepsilon_{\mu\nu\rho\sigma}\delta^{ab}\left(\frac{1}{2}\hat{c}^\nu B^{*\rho\sigma}_b + \frac{1}{6}\hat{c}^\nu\hat{c}^\rho Q^{*\sigma}_b - \frac{\mathrm{i}}{24}\hat{c}^\nu\hat{c}^\rho\hat{c}^\sigma R^*_b\right) \tag{4.14}$$

$$\hat{\nabla}_\mu\phi^a = \partial_\mu\phi^a + \frac{1}{4}\delta^{ab}\left(\mathrm{i}\xi\sigma_\mu\bar{\psi}^*_b + \mathrm{i}\psi^*_b\sigma_\mu\bar{\xi} - \hat{c}_\mu\phi^*_b\right) \tag{4.15}$$

$$\hat{\chi}_\alpha = \chi_\alpha - \frac{\mathrm{i}}{2}\hat{c}^\mu(\sigma_\mu\bar{\chi}^*)_\alpha \tag{4.16}$$

$$\hat{\nabla}_\mu\varphi = \nabla_\mu\varphi + \frac{\mathrm{i}}{2}\xi\sigma_\mu\bar{\chi}^* - \frac{1}{4}\hat{c}_\mu\bar{\varphi}^*. \tag{4.17}$$

$\hat{F}^{i(+)}$ and $\hat{F}^{i(-)}$ are the Lorentz irreducible parts of generalized Yang-Mills field strengths given by

$$\begin{aligned}\hat{F}^i_{\mu\nu} &= \frac{1}{2}\left(\hat{F}^{i(+)}_{\alpha\beta}\sigma^{\alpha\beta}_{\mu\nu} + \hat{F}^{i(-)}_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\dot{\alpha}\dot{\beta}}_{\mu\nu}\right)\\ &= F^i_{\mu\nu} + \delta^{ij}\left(\xi\sigma_{\mu\nu}\lambda^*_j + \bar{\lambda}^*_j\bar{\sigma}_{\mu\nu}\bar{\xi} + \frac{1}{3}\hat{c}_\mu A^*_{j\nu} - \frac{1}{3}\hat{c}_\nu A^*_{j\mu} - \frac{1}{6}\hat{c}_\mu\hat{c}_\nu C^*_j\right).\end{aligned} \tag{4.18}$$

5. Gauge covariant algebra

By construction the extended BRST transformations of the w's can be expressed solely in terms of the w's again, see eq. (4.1). As explained in [10], this is related to a graded commutator algebra which is realized on the $\hat{T}$'s. The cohomology of $s_{\rm ext}$ can be interpreted as the cohomology associated with this algebra (similiar to Lie algebra cohomology — in fact one may view it as a generalization of Lie algebra cohomology, see remark at the end of this section). We shall now discuss the extended BRST transformations of the w's and the corresponding algebra because these will be of crucial importance for the solution of the cohomological problem under study.

The extended BRST transformations of the w's can be directly obtained from the extended BRST transformations of the $w_{(0)}$'s:

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Lemma 4 ([11]) *The extended BRST transformations of the w's are given by $s_{\rm ext} w^I = r^I(w)$ with r^I the same function as in $s_{\rm ext} w^I_{(0)} = r^I(w_{(0)}) + O(1)$ where $O(1)$ collects all terms that are at least linear in the u's and v's.*

This lemma is very useful because it allows one to derive the extended BRST transformations of the w's without having to compute these variables and their extended BRST transformations explicitly (as remarked and demonstrated in section 4, the explicit structure of the w's is quite involved). The proof of the lemma was given in [11] and will not be repeated here. It is a consequence of the algorithm used to construct the w's (see lemma 3). For later purpose and to illustrate the lemma let us apply it to derive $s_{\rm ext}\hat{C}^i$. We start from $s_{\rm ext} C^i$ given by (2.15) and use that (2.8) gives $\partial_\mu C^i = {\rm i}\xi\sigma_\mu\bar{\lambda}^i - {\rm i}\lambda^i\sigma_\mu\bar{\xi} - \hat{c}^\nu\partial_{[\nu}A^i_{\mu]} + O(1)$. This yields

$$s_{\rm ext} C^i = \frac{1}{2} f_{kj}{}^i C^j C^k + {\rm i}\hat{c}^\mu \left(\xi\sigma_\mu\bar{\lambda}^i - \lambda^i\sigma_\mu\bar{\xi}\right) + \hat{c}^\mu\hat{c}^\nu\partial_\mu A^i_\nu + O(1)\,.$$

Applying now lemma 4 we conclude

$$s_{\rm ext}\hat{C}^i = \frac{1}{2} f_{kj}{}^i \hat{C}^j \hat{C}^k + {\rm i}\hat{c}^\mu \left(\xi\sigma_\mu\hat{\bar{\lambda}}^i - \hat{\lambda}^i\sigma_\mu\bar{\xi}\right) + \frac{1}{2}\hat{c}^\mu\hat{c}^\nu\hat{F}^i_{\mu\nu}\,. \tag{5.1}$$

When one verifies this result directly using (4.8) and the extended BRST transformations given in section 2, one finds that (5.1) actually contains no antifield dependent terms, i.e., all antifield dependent terms (coming from $\hat{\bar{\lambda}}^i$, $\hat{\lambda}^i$ and $\hat{F}^i_{\mu\nu}$) cancel out exactly on the right hand side of (5.1).

Analogously one can derive $s_{\rm ext}\hat{R}$ starting from $s_{\rm ext}R$ given by eq. (2.17) and using that (2.16) gives $\partial_\mu R = {\rm i}\hat{c}^\nu\partial_{[\nu}Q_{\mu]} - 2\xi\sigma_\mu\bar{\xi}\phi + O(1)$ and that (2.11) gives $\partial_{[\nu}Q_{\mu]} = -\frac{1}{2}\hat{c}^\rho\partial_{[\rho}B_{\mu\nu]} - \xi\sigma_{\mu\nu}\psi + \bar{\psi}\bar{\sigma}_{\mu\nu}\bar{\xi} + O(1)$. One obtains

$$s_{\rm ext}R = -2\hat{c}^\mu\xi\sigma_\mu\bar{\xi}\phi - {\rm i}\hat{c}^\mu\hat{c}^\nu\left(\xi\sigma_{\mu\nu}\psi - \bar{\psi}\bar{\sigma}_{\mu\nu}\bar{\xi}\right) - \frac{\rm i}{2}\hat{c}^\mu\hat{c}^\nu\hat{c}^\rho\partial_\mu B_{\nu\rho} + O(1)\,.$$

Using also (2.4) in the form $\partial_{[\mu}B_{\nu\rho]} = (1/3)\varepsilon_{\mu\nu\rho\sigma}H^\sigma$, one concludes

$$s_{\rm ext}\hat{R} = -2\hat{c}^\mu\xi\sigma_\mu\bar{\xi}\phi - {\rm i}\hat{c}^\mu\hat{c}^\nu\left(\xi\sigma_{\mu\nu}\hat{\psi} - \hat{\bar{\psi}}\bar{\sigma}_{\mu\nu}\bar{\xi}\right) - \frac{\rm i}{6}\hat{c}^\mu\hat{c}^\nu\hat{c}^\rho\varepsilon_{\mu\nu\rho\sigma}\hat{H}^\sigma. \tag{5.2}$$

Again, the antifield dependent terms on the right hand side cancel out exactly.

To derive the graded commutator algebra we proceed as in [10] and use that $\{w^I\}$ decomposes into subsets of variables with ghost numbers 2, 1 and 0, respectively. Those with ghost number 2 are the $\hat{R}^a$, those with ghost number 0 are the $\hat{T}^\tau$ and those with ghost number 1 are the $\hat{C}^i$, $\hat{c}^\mu$, $c^{\mu\nu}$, ξ^α, $\bar{\xi}^{\dot{\alpha}}$ which we denote collectively by $\mathcal{C}^M$ now,

$$\{\mathcal{C}^M\} = \{\hat{C}^i, \hat{c}^\mu, c^{\mu\nu}\ (\mu<\nu), \xi^\alpha, \bar{\xi}^{\dot{\alpha}}\}\,. \tag{5.3}$$

Since $s_{\rm ext}\hat{T}^\tau$ has ghost number 1 and since there are no w's with negative ghost numbers, we conclude from eq. (4.1) that $s_{\rm ext}\hat{T}^\tau$ is a linear combination of the $\mathcal{C}$'s with coefficients that are functions of the $\hat{T}$'s:

$$s_{\rm ext}\hat{T}^\tau = \mathcal{C}^M R^\tau_M(\hat{T})\,. \tag{5.4}$$

Moreover, since the $\hat{T}^\tau$ are independent variables, we can define operators Δ_M on the space of functions of the $\hat{T}$'s through

$$\Delta_M := R_M^\tau(\hat{T})\,\frac{\partial}{\partial \hat{T}^\tau}\,. \tag{5.5}$$

Using these operators, we can express the extended BRST transformation of any function of the $\hat{T}$'s according to

$$s_{\rm ext} f(\hat{T}) = \mathcal{C}^M \Delta_M f(\hat{T})\,. \tag{5.6}$$

By construction the Δ_M are graded derivations acting on the space of functions of the $\hat{T}$'s. For these graded derivations we introduce the following notation:

$$\{\Delta_M\} = \{\delta_i, \hat{\nabla}_\mu, l_{\mu\nu}, \mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\}\,, \tag{5.7}$$

so that

$$s_{\rm ext} f(\hat{T}) = \left(\hat{C}^i\delta_i + \hat{c}^\mu\hat{\nabla}_\mu + \frac{1}{2}c^{\mu\nu}l_{\mu\nu} + \xi^\alpha\mathcal{D}_\alpha + \bar{\xi}^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\right) f(\hat{T})\,. \tag{5.8}$$

The δ_i and $l_{\mu\nu}$ are linearly realized on the generalized tensor fields and represent the Lie algebras of the Yang-Mills gauge group and the Lorentz group as indicated by the indices of the $\hat{T}$'s, for instance:

$$\begin{aligned} \delta_i \hat{F}^{j(+)} &= -f_{ik}{}^j \hat{F}^{k(+)}\,, \qquad & \delta_i\phi &= 0\,, \qquad & \delta_i\varphi &= -T_i\varphi\,,\\ l_{\mu\nu}\hat{\lambda}^i_\alpha &= -(\sigma_{\mu\nu}\hat{\lambda}^i)_\alpha\,, & l_{\mu\nu}\phi &= 0\,, & l_{\mu\nu}\hat{\nabla}_\rho\phi &= \eta_{\rho\nu}\hat{\nabla}_\mu\phi - \eta_{\rho\mu}\hat{\nabla}_\nu\phi\,. \end{aligned}$$

In contrast, $\hat{\nabla}_\mu$, $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot\alpha}$ are nonlinearly realized, see comment b) below.

Lemma 5. *The graded commutator algebra of the graded derivations (5.7) reads*

$$\begin{aligned}
[\hat{\nabla}_\mu, \hat{\nabla}_\nu] &= -\hat{F}^i_{\mu\nu}\delta_i\\
[\mathcal{D}_\alpha, \hat{\nabla}_\mu] &= \mathrm{i}(\sigma_\mu\bar{\hat{\lambda}}^i)_\alpha\delta_i\,, \qquad [\bar{\mathcal{D}}_{\dot\alpha}, \hat{\nabla}_\mu] = -\mathrm{i}(\hat{\lambda}^i\sigma_\mu)_{\dot\alpha}\delta_i\\
\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} &= -2\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\hat{\nabla}_\mu\,, \qquad \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 0\\
[\delta_i, \hat{\nabla}_\mu] &= [\delta_i, \mathcal{D}_\alpha] = [\delta_i, \bar{\mathcal{D}}_{\dot\alpha}] = 0\\
[l_{\mu\nu}, \hat{\nabla}_\rho] &= \eta_{\rho\nu}\hat{\nabla}_\mu - \eta_{\rho\mu}\hat{\nabla}_\nu\,, \qquad [l_{\mu\nu}, \mathcal{D}_\alpha] = -\sigma_{\mu\nu\alpha}{}^\beta\mathcal{D}_\beta\,, \qquad [l_{\mu\nu}, \bar{\mathcal{D}}_{\dot\alpha}] = \bar{\sigma}_{\mu\nu}{}^{\dot\beta}{}_{\dot\alpha}\bar{\mathcal{D}}_{\dot\beta}\\
[\delta_i, \delta_j] &= f_{ij}{}^k\delta_k\,, \qquad [l_{\mu\nu}, l_{\rho\sigma}] = \eta_{\rho\nu}l_{\mu\sigma} + \eta_{\sigma\nu}l_{\rho\mu} - (\mu \leftrightarrow \nu)\,, \qquad [\delta_i, l_{\mu\nu}] = 0\,.
\end{aligned} \tag{5.9}$$

Comments.

(*a*) Notice that (5.9) is not a graded Lie algebra because $[\hat{\nabla}_\mu, \hat{\nabla}_\nu]$, $[\mathcal{D}_\alpha, \hat{\nabla}_\mu]$ and $[\bar{\mathcal{D}}_{\dot\alpha}, \hat{\nabla}_\mu]$ involve structure functions rather than structure constants. (5.9) may thus be rightly called a generalization of a (graded) Lie algebra and the cohomology of $s_{\rm ext}$ a generalization of (graded) Lie algebra cohomology.

(*b*) In order to avoid possible confusion, I stress that $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot\alpha}$ do *not* act in a superspace but are algebraically defined on the generalized tensor fields. The $\mathcal{D}_\alpha$-transformations of $\hat{\lambda}$, $\hat{F}^{(+)}$, ϕ, $\hat{\psi}$, $\hat{H}$, φ, $\hat{\chi}$ and their complex conjugates are spelled out explicitly in appendix A. From these one may derive the $\mathcal{D}_\alpha$-transformations of the first and higher order $\hat{\nabla}$-derivatives of these fields by means of the algebra (5.9) (e.g., $\mathcal{D}_\alpha\hat{\nabla}_\mu\bar{\hat{\lambda}}^i_{\dot\alpha} = ([\mathcal{D}_\alpha, \hat{\nabla}_\mu] + \hat{\nabla}_\mu\mathcal{D}_\alpha)\bar{\hat{\lambda}}^i_{\dot\alpha} = -\mathrm{i}(\sigma_\mu\bar{\hat{\lambda}}^j)_\alpha f_{jk}{}^i\bar{\hat{\lambda}}^k_{\dot\alpha}$). The $\bar{\mathcal{D}}_{\dot\alpha}$-transformations can be obtained from the $\mathcal{D}_\alpha$-transformations by complex conjugation.

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6. Computation of the cohomology

6.1 Cohomology for small ghost numbers

As a first cohomological result we shall now derive the cohomological groups $H^{g,4}(s_{\rm ext}|d)$ and $H^{g+4}(s_{\rm ext},\mathfrak{F})$ for $g<-1$. These groups can be straightforwardly obtained from corresponding cohomological groups of the so-called Koszul-Tate differential[6] δ modulo d. δ is part of $s_{\rm ext}$ and arises in the decomposition of $s_{\rm ext}$ according to the antifield number (af). The antifield number is defined according to standard conventions: R^*_a has antifield number 3, C^*_i and $Q^{*\mu}_a$ have antifield number 2, all other antifields have antifield number 1, the fields Φ^A and the constant ghosts have antifield number 0. The extended BRST differential decomposes into three pieces with antifield numbers -1, 0 and 1, respectively, with δ the piece with antifield number -1:

$$s_{\rm ext}=\delta+\gamma_{\rm ext}+s_{{\rm ext},1}\,,\qquad {\rm af}(\delta)=-1\,,\qquad {\rm af}(\gamma_{\rm ext})=0\,,\qquad {\rm af}(s_{{\rm ext},1})=1\,. \tag{6.1}$$

The δ-transformations of the fields, constant ghosts and antifields are:

$$\begin{aligned}
&\delta\Phi^A=\delta c^\mu=\delta\xi^\alpha=\delta\bar\xi^{\dot\alpha}=\delta c^{\mu\nu}=0\,,\\
&\delta\Phi^*_A=\frac{\hat\partial^R L}{\hat\partial\Phi^A}\qquad\text{for}\qquad \Phi^*_A\in\left\{A^{*\mu}_i,\lambda^{*\alpha}_i,\bar\lambda^{*\dot\alpha}_i,\phi^*_a,B^{*\mu\nu}_a,\psi^{*\alpha}_a,\bar\psi^{*\dot\alpha}_a,\varphi^*_s,\bar\varphi^{*s},\chi^{*\alpha}_s,\bar\chi^{*s\dot\alpha}\right\},\\
&\delta R^*_a={\rm i}\,\partial_\mu Q^{*\mu}_a\,,\qquad \delta Q^{*\mu}_a=2\partial_\nu B^{*\nu\mu}_a,\qquad \delta C^*_i=-\nabla_\mu A^{*\mu}_i+\sum_{\Phi^A\notin\{A^i_\mu\}}\Phi^*_A\delta_i\Phi^A\,.
\end{aligned}\tag{6.2}$$

Notice that the constant ghosts are inert to both δ and d. Therefore the cohomological groups $H(\delta|d)$ are the same as in the case of the non-extended BRST cohomology except that the representatives can depend arbitrarily on the constant ghosts.

Lemma 6. *Cohomology $H^4_k(\delta|d)$ for antifield numbers $k>1$:*

$$\begin{aligned}
&\delta\omega^4_k+d\omega^3_{k-1}=0\;\Rightarrow\;\omega^4_k\sim\begin{cases}0 & \text{for } k>3\\ k^a(c,\xi,\bar\xi)R^*_a\,d^4x & \text{for } k=3\\ [k^{i_{\rm f}}(c,\xi,\bar\xi)C^*_{i_{\rm f}}+k^{[ab]}(c,\xi,\bar\xi)f_{ab}]\,d^4x & \text{for } k=2\,,\end{cases}\\
&f_{ab}=Q^{*\mu}_{[a}H_{b]\mu}-\frac12\varepsilon_{\mu\nu\rho\sigma}B^{*\mu\nu}_aB^{*\rho\sigma}_b\,,
\end{aligned}\tag{6.3}$$

where $\sim$ is equivalence in $H(\delta|d)$ ($\omega^4_k\sim\omega^4_k+\delta\omega^4_{k+1}+d\omega^3_k$), $i_{\rm f}$ runs over those abelian elements of $\mathfrak{g}_{\rm YM}$ under which all matter fields are uncharged, the $k(c,\xi,\bar\xi)$'s are arbitrary functions of the constant ghosts, and we used the notation $H_{a\mu}:=\delta_{ab}H^b_\mu$. The 4-forms $R^*_ad^4x$, $C^*_{i_{\rm f}}d^4x$ and $f_{ab}d^4x$, $a<b$, are nontrivial and inequivalent in $H(\delta|d)$:

$$k^a(c,\xi,\bar\xi)R^*_a\sim0\;\Rightarrow\;k^a=0\,;\tag{6.4}$$

$$\left[k^{i_{\rm f}}(c,\xi,\bar\xi)C^*_{i_{\rm f}}+k^{[ab]}(c,\xi,\bar\xi)f_{ab}\right]d^4x\sim0\;\Rightarrow\;k^{i_{\rm f}}=k^{[ab]}=0\,.\tag{6.5}$$

[6] We are dealing here with the standard Koszul-Tate differential, trivially extended to the constant ghosts. It must not be confused with the extended Koszul-Tate differential introduced in [13] which acts also on “global antifields” conjugate to the constant ghosts.

Using lemma 6, it is straightforward to prove the following result:

Lemma 7. *Cohomology $H^{g,4}(s_{\rm ext}|d)$ for ghost numbers $g<-1$:*

$$
s_{\rm ext}\omega^{g,4}+d\omega^{g-1,3}=0\ \Rightarrow\ \omega^{g,4}\sim\begin{cases}0 & \textit{for}\quad g<-3\\ k^a R^*_a\,d^4x & \textit{for}\quad g=-3\\ (k^{i_{\rm f}}C^*_{i_{\rm f}}+k^{[ab]}f'_{ab})\,d^4x & \textit{for}\quad g=-2,\end{cases}
$$

$$
f'_{ab}=Q^{*\mu}_{[a}\left(H_{b]\mu}+\psi^*_{b]}\sigma_\mu\bar\xi-\xi\sigma_\mu\bar\psi^*_{b]}\right)-\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}B^{*\mu\nu}_a B^{*\rho\sigma}_b+R^*_{[a}\left(\bar\psi_{b]}\bar\xi-\xi\psi_{b]}\right), \tag{6.6}
$$

where $\sim$ is equivalence in $H(s_{\rm ext}|d)$ ($\omega^{g,4}\sim\omega^{g,4}+s_{\rm ext}\omega^{g-1,4}+d\omega^{g,3}$), $i_{\rm f}$ runs over those abelian elements of $\mathfrak{g}_{\rm YM}$ under which all matter fields are uncharged, and k^a, $k^{i_{\rm f}}$ and $k^{[ab]}$ are complex numbers. The 4-forms $R^*_a d^4x$, $C^*_{i_{\rm f}}d^4x$ and $f'_{ab}d^4x$, $a<b$, are nontrivial and inequivalent in $H(s_{\rm ext}|d)$:

$$
k^a R^*_a\,d^4x\sim 0\ \Rightarrow\ k^a=0\,; \tag{6.7}
$$

$$
(k^{i_{\rm f}}C^*_{i_{\rm f}}+k^{[ab]}f'_{ab})\,d^4x\sim 0\ \Rightarrow\ k^{i_{\rm f}}=k^{[ab]}=0\,. \tag{6.8}
$$

Using the relation between $H^{g,4}(s_{\rm ext}|d)$ and $H^{g+4}(s_{\rm ext},\mathfrak{F})$ (see section 3) it is now immediate to derive $H^g(s_{\rm ext},\mathfrak{F})$ for ghost numbers smaller than 3:

Lemma 8. *Cohomology $H^g(s_{\rm ext},\mathfrak{F})$ for ghost numbers $g<3$:*

$$
s_{\rm ext}f^g=0\ \Rightarrow\ f^g\sim\begin{cases}0 & \textit{for}\quad g<0\\ k & \textit{for}\quad g=0\\ k_a\tilde H^a & \textit{for}\quad g=1\\ k_{i_{\rm f}}\tilde F^{i_{\rm f}}+\tfrac{1}{2}k_{[ab]}\tilde H^a\tilde H^b & \textit{for}\quad g=2\,,\end{cases}
$$

$$
\begin{aligned}
\tilde H^a&={\rm i}\left(\bar{\hat\psi}{}^a\bar\xi-\xi\hat\psi^a\right)+\hat c^\mu\hat H^a_\mu,\\
\tilde F^{i_{\rm f}}&=\hat c^\mu\left(\xi\sigma_\mu\bar{\hat\lambda}{}^{i_{\rm f}}+\hat\lambda^{i_{\rm f}}\sigma_\mu\bar\xi\right)+\frac{1}{4}\hat c^\mu\hat c^\nu\varepsilon_{\mu\nu\rho\sigma}\hat F^{i_{\rm f}\rho\sigma},
\end{aligned} \tag{6.9}
$$

where $\sim$ is equivalence in $H(s_{\rm ext},\mathfrak{F})$ ($f^g\sim f^g+s_{\rm ext}f^{g-1}$ with $f^g,f^{g-1}\in\mathfrak{F}$), $i_{\rm f}$ runs over those abelian elements of $\mathfrak{g}_{\rm YM}$ under which all matter fields are uncharged, and k, k_a, $k_{i_{\rm f}}$ and $k_{[ab]}$ are complex numbers. The cocycles 1, $\tilde H^a$, $\tilde F^{i_{\rm f}}$ and $\tilde H^a\tilde H^b$, $a<b$, are nontrivial and inequivalent in $H(s_{\rm ext},\mathfrak{F})$:

$$
k\sim 0\ \Rightarrow\ k=0\,; \tag{6.10}
$$

$$
k_a\tilde H^a\sim 0\ \Rightarrow\ k_a=0\,; \tag{6.11}
$$

$$
k_{i_{\rm f}}\tilde F^{i_{\rm f}}+\frac{1}{2}k_{[ab]}\tilde H^a\tilde H^b\sim 0\ \Rightarrow\ k_{i_{\rm f}}=k_{[ab]}=0\,. \tag{6.12}
$$

Comment. In n-dimensional theories, the representatives of $H^n_{n-p}(\delta|d)$ are related through descent equations for δ and d to conserved local p-forms (i.e., p-forms which do not depend on antifields and satisfy $d\omega^p\approx 0$) representing the so-called characteristic cohomology of the field equations [5]. Therefore one can conclude from lemmas 6 and 7 that

$H^{-3,4}(s_{\rm ext}|d)$ and $H^{-2,4}(s_{\rm ext}|d)$ are isomorphic to the characteristic cohomology in form-degrees $p=1$ and $p=2$, respectively, and that the latter is represented by the 1-forms $\star dB^a$ and the 2-forms $\star dA^{i_{\rm f}}$ and $(\star dB^a)(\star dB^b)$ where $\star$ denotes Hodge-dualization and $A^{i_{\rm f}}=dx^\mu A^{i_{\rm f}}_\mu$ and $B^a=(1/2)dx^\mu dx^\nu B^a_{\mu\nu}$. We also observe that $H^0(s_{\rm ext},\mathfrak{F})$, $H^1(s_{\rm ext},\mathfrak{F})$ and $H^2(s_{\rm ext},\mathfrak{F})$ are isomorphic to the characteristic cohomology in form-degrees $p=0$, $p=1$ and $p=2$, respectively. We shall see that a similar result does not hold for $p=3$, cf. section 8.

6.2 Elimination of trivial pairs

To compute $H^g(s_{\rm ext},\mathfrak{F})$ for ghost numbers $g\geq 3$ we use the jet-coordinates u^ℓ,v^ℓ,w^I given in section 4. Thanks to eq. (4.1), the jet-variables u^ℓ and v^ℓ form trivial pairs in the terminology of [10] and drop from (the nontrivial part of) $H(s_{\rm ext},\mathfrak{F})$:

Lemma 9. *$H(s_{\rm ext},\mathfrak{F})$ is isomorphic to the cohomology of $s_{\rm ext}$ in the space $\mathfrak{W}$ of local functions of the variables w^I listed in eq. (4.6):*

$$H(s_{\rm ext},\mathfrak{F})\simeq H(s_{\rm ext},\mathfrak{W})\,,\qquad \mathfrak{W}=\{f(w)\}\,.$$

6.3 Decomposition of the cohomological problem

To compute $H(s_{\rm ext},\mathfrak{W})$, we use the counting operator N for the variables $\hat{C}^i$, $c^{\mu\nu}$ and $\hat{R}^a$ as a filtration,

$$N=\hat{C}^i\,\frac{\partial}{\partial\hat{C}^i}+\frac{1}{2}c^{\mu\nu}\,\frac{\partial}{\partial c^{\mu\nu}}+\hat{R}^a\,\frac{\partial}{\partial\hat{R}^a}\,.$$

$s_{\rm ext}$ decomposes in $\mathfrak{W}$ into three parts with N-degrees 1, 0 and -1 which we denote by $s_{\rm lie}$, $s_{\rm susy}$ and $s_{\rm curv}$, respectively:

$$\begin{aligned}
s_{\rm ext}f(w) &= (s_{\rm lie}+s_{\rm susy}+s_{\rm curv})f(w)\\
[N,s_{\rm lie}] &= s_{\rm lie}\,,\qquad [N,s_{\rm susy}]=0\,,\qquad [N,s_{\rm curv}]=-s_{\rm curv}\\
s_{\rm lie} &= \frac{1}{2}f_{kj}{}^i\hat{C}^j\hat{C}^k\,\frac{\partial}{\partial\hat{C}^i}-c_\nu{}^\rho c_\rho{}^\mu\,\frac{\partial}{\partial c_\nu{}^\mu}+\\
&\quad +c_\nu{}^\mu\hat{c}^\nu\,\frac{\partial}{\partial\hat{c}^\mu}+\frac{1}{2}c^{\mu\nu}(\xi\sigma_{\mu\nu})^\alpha\,\frac{\partial}{\partial\xi^\alpha}-\frac{1}{2}c^{\mu\nu}(\bar\sigma_{\mu\nu}\bar\xi)^{\dot\alpha}\,\frac{\partial}{\partial\bar\xi^{\dot\alpha}}+\\
&\quad +\left(\hat{C}^i\delta_i+\frac{1}{2}c^{\mu\nu}l_{\mu\nu}\right)\hat{T}^\tau\,\frac{\partial}{\partial\hat{T}^\tau}\\
s_{\rm susy} &= 2{\rm i}\xi\sigma^\mu\bar\xi\,\frac{\partial}{\partial\hat{c}^\mu}+(\hat{c}^\mu\hat\nabla_\mu+\xi^\alpha\mathcal{D}_\alpha+\bar\xi^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha})\hat{T}^\tau\,\frac{\partial}{\partial\hat{T}^\tau}\\
s_{\rm curv} &= \mathcal{F}^i\,\frac{\partial}{\partial\hat{C}^i}+\mathcal{H}^a\,\frac{\partial}{\partial\hat{R}^a}
\end{aligned}\tag{6.13}$$

where we introduced

$$\mathcal{F}^i={\rm i}\hat{c}^\mu(\xi\sigma_\mu\hat{\bar\lambda}^i-\hat\lambda^i\sigma_\mu\bar\xi)+\frac{1}{2}\hat{c}^\mu\hat{c}^\nu\hat{F}^i_{\mu\nu}\tag{6.14}$$

$$\mathcal{H}^a=-2\hat{c}^\mu\xi\sigma_\mu\bar\xi\phi^a-{\rm i}\hat{c}^\mu\hat{c}^\nu(\xi\sigma_{\mu\nu}\hat\psi^a-\hat{\bar\psi}^a\bar\sigma_{\mu\nu}\bar\xi)-\frac{\rm i}{6}\hat{c}^\mu\hat{c}^\nu\hat{c}^\rho\varepsilon_{\mu\nu\rho\sigma}\hat{H}^{a\sigma}\,.\tag{6.15}$$

$s_{\rm ext}^2=0$ decomposes under the N-degree according to:

$$s_{\rm lie}^2=s_{\rm curv}^2=\{s_{\rm lie},s_{\rm susy}\}=\{s_{\rm curv},s_{\rm susy}\}=0,\qquad \{s_{\rm lie},s_{\rm curv}\}+s_{\rm susy}^2=0\,. \tag{6.16}$$

Let us denote by f_m the piece with N-degree m of a function $f\in\mathfrak{W}$, and by $\overline{m}$ and $\underline{m}$ the highest and lowest N-degrees contained in f, respectively.[7] The cocycle condition $s_{\rm ext}f(w)=0$ in $H(s_{\rm ext},\mathfrak{W})$ decomposes into:

$$s_{\rm lie}f_{\overline{m}}=0 \tag{6.17}$$

$$s_{\rm susy}f_{\overline{m}}+s_{\rm lie}f_{\overline{m}-1}=0 \tag{6.18}$$

$$s_{\rm curv}f_{\overline{m}}+s_{\rm susy}f_{\overline{m}-1}+s_{\rm lie}f_{\overline{m}-2}=0 \tag{6.19}$$

$$\vdots$$

$$s_{\rm curv}f_{\underline{m}}=0\,. \tag{6.20}$$

6.4 Lie algebra cohomology

Eq. (6.17) shows that $f_{\overline{m}}$ is a cocycle of $s_{\rm lie}$. We can assume that it is not a coboundary of $s_{\rm lie}$ because otherwise we could remove it from the $s_{\rm ext}$-cocycle $f(w)$ without changing the cohomology class of the latter (if $f_{\overline{m}}=s_{\rm lie}h_{\overline{m}-1}$, one replaces f with $f-s_{\rm ext}h_{\overline{m}-1}$ which is equivalent to f in $H(s_{\rm ext},\mathfrak{W})$). Hence, $f_{\overline{m}}$ can be assumed to be a nontrivial representative of the cohomology of $s_{\rm lie}$ in $\mathfrak{W}$ which we denote by $H(s_{\rm lie},\mathfrak{W})$. This cohomology is well-known: it is the Lie algebra cohomology of $\mathfrak{g}=\mathfrak{g}_{\rm YM}+\mathfrak{so}(1,3)$, with $\mathfrak{g}$ represented on the local functions of $\hat c^\mu$, ξ^α, $\bar\xi^{\dot\alpha}$ and $\hat T^\tau$. This cohomology is generated by so-called primitive elements θ_r constructed of the $\hat C^i$ and $c^{\mu\nu}$, $\mathfrak{g}$-invariant functions of the $\hat c^\mu$, ξ^α, $\bar\xi^{\dot\alpha}$ and $\hat T^\tau$, and linearly independent polynomials in the $\hat R^a$ (notice that the $\hat R^a$ are inert to $s_{\rm lie}$). The θ's correspond one-to-one to the independent Casimir operators of $\mathfrak{g}$. The index r of the θ's runs thus from 1 to $\mathrm{rank}(\mathfrak{g})=\mathrm{rank}(\mathfrak{g}_{\rm YM})+2$. The θ's of $\mathfrak{g}_{\rm YM}$ can be constructed by means of suitable matrix representations $\{T_i^{(r)}\}$ of $\mathfrak{g}_{\rm YM}$ (the superscript (r) of $T_i^{(r)}$ indicates that the respective representation may depend on the value of r):

$$\theta_r=(-)^{m(r)-1}\frac{m(r)!(m(r)-1)!}{(2m(r)-1)!}\,\mathrm{tr}_r(\hat C^{2m(r)-1})\,,\quad \hat C=\hat C^iT_i^{(r)}\,,\quad r=1,\dots,\mathrm{rank}(\mathfrak{g}_{\rm YM})\,, \tag{6.21}$$

where $m(r)$ is the order of the corresponding Casimir operator of $\mathfrak{g}_{\rm YM}$. The θ's with $m(r)=1$ can be taken to coincide with the abelian $\hat C$'s by choosing $T_i^{(r)}=1$ for one of the abelian elements of $\mathfrak{g}_{\rm YM}$ and $T_i^{(r)}=0$ for all other elements. We denote the abelian $\hat C$'s by $\hat C^{i_A}$:

$$\{\theta_r:\ m(r)=1\}=\{\hat C^{i_A}\}=\{\text{abelian }\hat C\text{'s}\}. \tag{6.22}$$

$\mathfrak{so}(1,3)$ contributes two additional θ's with $m(r)=2$ ("Lorentz-θ's"):

$$\theta_{\mathrm{rank}(\mathfrak{g}_{\rm YM})+1}=c_\mu{}^\nu c_\nu{}^\rho c_\rho{}^\mu\,,\qquad \theta_{\mathrm{rank}(\mathfrak{g}_{\rm YM})+2}=\varepsilon_{\mu\nu\rho\sigma}c^{\mu\nu}c^{\rho\lambda}c_\lambda{}^\sigma\,. \tag{6.23}$$

[7] We can always assume that $\overline{m}$ is finite because it is bounded by the ghost number of f (recall that there are no w's with negative ghost number).

We denote the space of $\mathfrak{g}$-invariant local functions of the $\hat{c}^\mu$, ξ^α, $\bar{\xi}^{\dot{\alpha}}$ and $\hat{T}^\tau$ by $\mathfrak{T}_{\rm inv}$,

$$\mathfrak{T}_{\rm inv} = \{f(\hat{c},\xi,\bar{\xi},\hat{T}):\ \delta_i f(\hat{c},\xi,\bar{\xi},\hat{T}) = 0,\ l_{\mu\nu} f(\hat{c},\xi,\bar{\xi},\hat{T}) = 0\}. \tag{6.24}$$

$H(s_{\rm lie},\mathfrak{W})$ can now be described as follows:

$$s_{\rm lie} f(w) = 0 \qquad \Leftrightarrow \qquad f(w) = s_{\rm lie} h(w) + f^\Gamma P_\Gamma(\theta,\hat{R})\,, \quad f^\Gamma \in \mathfrak{T}_{\rm inv}\,; \tag{6.25}$$
$$f^\Gamma P_\Gamma(\theta,\hat{R}) = s_{\rm lie} h(w)\,, \quad f^\Gamma \in \mathfrak{T}_{\rm inv} \qquad \Rightarrow \qquad f^\Gamma = 0\,, \tag{6.26}$$

where $\{P_\Gamma(\theta,\hat{R})\}$ is a basis of the monomials in the θ_r and $\hat{R}^a$. As mentioned above, this expresses the Lie algebra cohomology of $\mathfrak{g}$, with representation space given by the local functions of the $\hat{c}^\mu$, ξ^α, $\bar{\xi}^{\dot{\alpha}}$ and $\hat{T}^\tau$. It is a well-known result, see, e.g., section 8 of [7], and implies directly the following:

Lemma 10. *The piece with highest N-degree of a representative f of $H(s_{\rm ext},\mathfrak{W})$ can be assumed to be of the form*

$$f_{\overline{m}} = f^{\Gamma_{\overline{m}}} P_{\Gamma_{\overline{m}}}(\theta,\hat{R})\,, \qquad s_{\rm susy} f^{\Gamma_{\overline{m}}} = 0, \qquad f^{\Gamma_{\overline{m}}} \neq s_{\rm susy} g^{\Gamma_{\overline{m}}}\,, \qquad f^{\Gamma_{\overline{m}}}, g^{\Gamma_{\overline{m}}} \in \mathfrak{T}_{\rm inv}\,, \tag{6.27}$$

where $\{P_{\Gamma_m}(\theta,\hat{R})\} = \{P_\Gamma(\theta,\hat{R}) : N P_\Gamma(\theta,\hat{R}) = m P_\Gamma(\theta,\hat{R})\}$.

6.5 Supersymmetry algebra cohomology

Eqs. (6.27) shows that the functions $f^{\Gamma_{\overline{m}}}$ are representatives of the cohomology of $s_{\rm susy}$ in the space $\mathfrak{T}_{\rm inv}$. We denote this cohomology by $H(s_{\rm susy},\mathfrak{T}_{\rm inv})$. It is indeed well-defined because $s_{\rm susy}$ squares to zero on all functions in $\mathfrak{T}_{\rm inv}$: according to (6.16) one has $s_{\rm susy}^2 f = -\{s_{\rm lie}, s_{\rm curv}\} f$ which vanishes for $f \in \mathfrak{T}_{\rm inv}$ because one has $s_{\rm lie} f = 0$ and $s_{\rm curv} f = 0$ for $f \in \mathfrak{T}_{\rm inv}$ by definition of $\mathfrak{T}_{\rm inv}$. In particular one has

$$f \in \mathfrak{T}_{\rm inv} \qquad \Rightarrow \qquad s_{\rm susy} f = s_{\rm ext} f\,, \tag{6.28}$$

and thus

$$H(s_{\rm susy},\mathfrak{T}_{\rm inv}) = H(s_{\rm ext},\mathfrak{T}_{\rm inv})\,. \tag{6.29}$$

The following lemma describes this cohomology and is a key result of the computation.

Lemma 11. *Cohomology $H(s_{\rm susy},\mathfrak{T}_{\rm inv})$:*

(i) *The general solution of the cocycle condition in $H^g(s_{\rm susy},\mathfrak{T}_{\rm inv})$ for the various ghost numbers g is:*

$$s_{\rm susy} f^g = 0,\ f^g \in \mathfrak{T}_{\rm inv} \ \Leftrightarrow\ f^g \sim \begin{cases} k & \textit{for} \quad g = 0 \\ k_a \tilde{H}^a & \textit{for} \quad g = 1 \\ k_{i_{\rm f}} \tilde{F}^{i_{\rm f}} + \frac{1}{2} k_{[ab]} \tilde{H}^a \tilde{H}^b + k_{i_A} \mathcal{F}^{i_A} & \textit{for} \quad g = 2 \\ \mathcal{O}R(\hat{T}) & \textit{for} \quad g = 3 \\ \mathcal{P}\Omega_1 + \bar{\mathcal{P}}\bar{\Omega}_2 & \textit{for} \quad g = 4 \\ 0 & \textit{for} \quad g \geq 5 \end{cases}$$

$$
\begin{aligned}
&\mathcal{O} = 4\hat{c}^{\mu}\xi\sigma_{\mu}\bar{\xi} - 4\,\Xi_{\mu\nu}(\xi\sigma^{\mu\nu}\mathcal{D} + \bar{\xi}\bar{\sigma}^{\mu\nu}\bar{\mathcal{D}}) - \frac{\mathrm{i}}{2}\Xi_{\mu}\sigma^{\mu}_{\alpha\dot{\alpha}}[\mathcal{D}^{\alpha}, \bar{\mathcal{D}}^{\dot{\alpha}}] \\
&\mathcal{P} = -16\,\Xi_{\mu\nu}\,\bar{\xi}\bar{\sigma}^{\mu\nu}\bar{\xi} - 4\mathrm{i}\,\Xi_{\mu}\bar{\xi}\bar{\sigma}^{\mu}\mathcal{D} + \Xi\,\mathcal{D}^2 \\
&\Xi_{\mu\nu} = -\frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}\hat{c}^{\rho}\hat{c}^{\sigma}\,, \qquad \Xi_{\mu} = -\frac{1}{6}\varepsilon_{\mu\nu\rho\sigma}\hat{c}^{\nu}\hat{c}^{\rho}\hat{c}^{\sigma}\,, \qquad \Xi = -\frac{1}{24}\varepsilon_{\mu\nu\rho\sigma}\hat{c}^{\mu}\hat{c}^{\nu}\hat{c}^{\rho}\hat{c}^{\sigma} \\
&R(\hat{T}) \in \mathfrak{T}_{\mathrm{inv}}\,, \qquad \mathcal{D}^2 R(\hat{T}) = \bar{\mathcal{D}}^2 R(\hat{T}) = 0 \\
&\Omega_i = A_i(\varphi, \hat{\lambda}) + \bar{\mathcal{D}}^2 B_i(\hat{T})\,, \qquad A_i(\varphi, \hat{\lambda}), B_i(\hat{T}) \in \mathfrak{T}_{\mathrm{inv}} \qquad (i = 1, 2)\,, \qquad (6.30)
\end{aligned}
$$

where $\sim$ is equivalence in $H(s_{\mathrm{susy}}, \mathfrak{T}_{\mathrm{inv}})$ ($f^g \sim f^g + s_{\mathrm{susy}} f^{g-1}$ with $f^g, f^{g-1} \in \mathfrak{T}_{\mathrm{inv}}$), i_A runs over all abelian elements of $\mathfrak{g}_{\mathrm{YM}}$ and $\mathcal{F}^{i_A}$ are the abelian $\mathcal{F}$'s (6.14), k, k_a, $k_{i_{\mathrm{f}}}$, k_{ab} and k_{i_A} are complex numbers, we used the notation $\mathcal{D}^2 = \mathcal{D}^{\alpha}\mathcal{D}_{\alpha}$, $\bar{\mathcal{D}}^2 = \bar{\mathcal{D}}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}$, $[\mathcal{D}^{\alpha}, \bar{\mathcal{D}}^{\dot{\alpha}}] = \mathcal{D}^{\alpha}\bar{\mathcal{D}}^{\dot{\alpha}} - \bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{D}^{\alpha}$ and other notation as in lemma 8. The functions $A_i(\varphi, \hat{\lambda})$ depend only on the undifferentiated φ^s and $\hat{\lambda}^i$ but not on any (generalized covariant) derivatives thereof.

(ii) The cocycles 1, $\tilde{H}^a$, $\tilde{F}^{i_{\mathrm{f}}}$, $\tilde{H}^a\tilde{H}^b$, $a < b$, and $\mathcal{F}^{i_A}$ are nontrivial and inequivalent in $H(s_{\mathrm{susy}}, \mathfrak{T}_{\mathrm{inv}})$, a cocycle $\mathcal{O}R(\hat{T})$ is trivial in $H(s_{\mathrm{susy}}, \mathfrak{T}_{\mathrm{inv}})$ iff $R(\hat{T}) = \mathcal{D}\Omega_1 + \bar{\mathcal{D}}\bar{\Omega}_2$ for some functions $\Omega_i^{\alpha} = A_i^{\alpha}(\varphi, \hat{\lambda}) + \bar{\mathcal{D}}^2 B_i^{\alpha}(\hat{T})$ ($i = 1, 2$), and a cocycle $\mathcal{P}\Omega_1 + \bar{\mathcal{P}}\bar{\Omega}_2$ is trivial in $H(s_{\mathrm{susy}}, \mathfrak{T}_{\mathrm{inv}})$ iff both $\Omega_1 = \bar{\mathcal{D}}^2 X(\hat{T})$ and $\bar{\Omega}_2 = -\mathcal{D}^2 X(\hat{T})$ for some (the same!) function $X(\hat{T}) \in \mathfrak{T}_{\mathrm{inv}}$:[8]

$$
\begin{aligned}
&k \sim 0 \qquad \Leftrightarrow \qquad k = 0\,; \\
&k_a\tilde{H}^a \sim 0 \qquad \Leftrightarrow \qquad k_a = 0\,; \\
&k_{i_{\mathrm{f}}}\tilde{F}^{i_{\mathrm{f}}} + \frac{1}{2}k_{[ab]}\tilde{H}^a\tilde{H}^b + k_{i_A}\mathcal{F}^{i_A} \sim 0 \qquad \Leftrightarrow \qquad k_{i_{\mathrm{f}}} = k_{[ab]} = k_{i_A} = 0\,; \\
&\mathcal{O}R(\hat{T}) \sim 0 \qquad \Leftrightarrow \qquad R(\hat{T}) = \mathcal{D}\Omega_1 + \bar{\mathcal{D}}\bar{\Omega}_2, \qquad \Omega_i^{\alpha} = A_i^{\alpha}(\varphi, \hat{\lambda}) + \bar{\mathcal{D}}^2 B_i^{\alpha}(\hat{T})\,; \\
&\mathcal{P}\Omega_1 + \bar{\mathcal{P}}\bar{\Omega}_2 \sim 0 \qquad \Leftrightarrow \qquad [\Omega_1 = \bar{\mathcal{D}}^2 X(\hat{T}) \wedge \bar{\Omega}_2 = -\mathcal{D}^2 X(\hat{T})]\,. \qquad (6.31)
\end{aligned}
$$

Comments.

(*a*) In section 6.1 we found that the cohomology groups $H^0(s_{\mathrm{ext}}, \mathfrak{F})$, $H^1(s_{\mathrm{ext}}, \mathfrak{F})$ and $H^2(s_{\mathrm{ext}}, \mathfrak{F})$ are isomorphic to the characteristic cohomology in form-degrees 0, 1 and 2, respectively (see comments at the end of that section). A similar result holds for $H^0(s_{\mathrm{susy}}, \mathfrak{T}_{\mathrm{inv}})$, $H^1(s_{\mathrm{susy}}, \mathfrak{T}_{\mathrm{inv}})$ and $H^2(s_{\mathrm{susy}}, \mathfrak{T}_{\mathrm{inv}})$: they are isomorphic to the gauge invariant characteristic cohomology (characteristic cohomology in the space of gauge invariant local forms) in form-degrees 0, 1 and 2, respectively. Indeed one can show as in [14, 15] that the latter is in form-degrees 0, 1 and 2 represented by:

$$
\begin{aligned}
&p = 0: \qquad \omega^0 = k\,; &(6.32)\\
&p = 1: \qquad \omega^1 = k_a \star dB^a\,; &(6.33)\\
&p = 2: \qquad \omega^2 = k_{i_{\mathrm{f}}} \star dA^{i_{\mathrm{f}}} + \frac{1}{2}k_{[ab]}(\star dB^a)(\star dB^b) + k_{i_A} dA^{i_A}\,. &(6.34)
\end{aligned}
$$

[8] $\Omega_1 = \bar{\mathcal{D}}^2 X \wedge \bar{\Omega}_2 = -\mathcal{D}^2 X$ is equivalent to $A_1 = 0 \wedge A_2 = 0 \wedge \bar{\mathcal{D}}^2 B_1 = \bar{\mathcal{D}}^2 X \wedge \mathcal{D}^2 B_2 = -\mathcal{D}^2 X$ because no function $A(\varphi, \hat{\lambda})$ is of the form $\bar{\mathcal{D}}^2(\dots)$.

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(*b*) Notice that the lemma characterizes the cohomology in ghost number 3 through functions $R(\hat{T}) \in \mathfrak{T}_{\rm inv}$ which satisfy $\mathcal{D}^2 R(\hat{T}) = \bar{\mathcal{D}}^2 R(\hat{T}) = 0$ and are determined up to contributions of the form $\mathcal{D}\Omega_1 + \bar{\mathcal{D}}\bar{\Omega}_2$ with $\Omega_i^\alpha = A_i^\alpha(\varphi, \hat{\lambda}) + \bar{\mathcal{D}}^2 B_i^\alpha(\hat{T})$ [such contributions can be dropped because of part (ii) of the lemma]. I have not determined the general solution to these conditions (which appears to be a rather involved problem) but would like to add the following comments concerning this result.

The simplest nontrivial functions $R(\hat{T})$ are complex numbers, i.e., $R(\hat{T}) = k \in \mathbb{C}$ with $k \neq 0$. They yield field independent representatives with ghost number 3 given by

$$4k\, \hat{c}^\mu \xi \sigma_\mu \bar{\xi}\,. \tag{6.35}$$

All other representatives $\mathcal{O}R(\hat{T})$ contain gauge invariant conserved currents j^μ given by the antifield independent parts of $(1/2)\bar{\sigma}^\mu_{\alpha\dot{\alpha}}[\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}]R(\hat{T})$,

$$j^\mu = \Big[\frac{1}{2}\bar{\sigma}^\mu_{\alpha\dot{\alpha}}[\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}]R(\hat{T})\Big]_{\Phi^*=0}\,. \tag{6.36}$$

That these currents are indeed conserved can be directly verified by means of the algebra (5.9) which implies, for all δ_i-invariant functions $f(\hat{T})$:

$$\delta_i f(\hat{T}) = 0 \qquad \Rightarrow \qquad [\mathcal{D}^2, \bar{\mathcal{D}}^2] f(\hat{T}) = -4\mathrm{i}\hat{\nabla}_{\alpha\dot{\alpha}}[\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}] f(\hat{T})\,. \tag{6.37}$$

Specializing this formula to $R(\hat{T})$, it yields $0 = \hat{\nabla}_{\alpha\dot{\alpha}}[\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}]R(\hat{T})$ because of $\mathcal{D}^2 R(\hat{T}) = \bar{\mathcal{D}}^2 R(\hat{T}) = 0$. Owing to (4.7) and the gauge invariance of j^μ (which follows from $R(\hat{T}) \in \mathfrak{T}_{\rm inv}$), this implies that j^μ is indeed conserved:

$$0 \approx \nabla_\mu j^\mu = \partial_\mu j^\mu\,. \tag{6.38}$$

j^μ gives thus rise to a cocycle $\omega^3 = (1/6) dx^\mu dx^\nu dx^\rho \varepsilon_{\mu\nu\rho\sigma} j^\sigma$ of the gauge invariant characteristic cohomology in form-degree 3. One can show that this 3-form is trivial in the gauge invariant characteristic cohomology iff $\mathcal{O}R(\hat{T})$ is in $H(s_{\rm susy}, \mathfrak{T}_{\rm inv})$ equivalent to a function (6.35).[9] Hence, except for the field independent representatives (6.35), nontrivial functions $\mathcal{O}R(\hat{T})$ correspond to representatives of the gauge invariant characteristic cohomology in form-degree 3. However, this correspondence is not one-to-one as it is not surjective: there are representatives of the gauge invariant characteristic cohomology in form-degree 3 which do not have a counterpart in $H^3(s_{\rm susy}, \mathfrak{T}_{\rm inv})$. In particular the Noether currents of supersymmetry and Poincaré symmetry (and also those of other conformal symmetries) do not correspond to representatives of $H^3(s_{\rm susy}, \mathfrak{T}_{\rm inv})$ as one can already deduce from the fact

[9] Using part (ii) of lemma 11, the algebra (5.9), and eq. (4.7), it is straightforward to verify that $\mathcal{O}[R(\hat{T}) - k] \sim 0$ implies $j^\mu \approx \partial_\nu S^{\nu\mu}$ with $S^{\nu\mu} = 4\mathrm{i}(\mathcal{D}\sigma^{\nu\mu}\Omega_1 + \bar{\mathcal{D}}\bar{\sigma}^{\mu\nu}\bar{\Omega}_2)$ for some gauge invariant $\Omega_i^\alpha = A_i^\alpha(\varphi, \hat{\lambda}) + \bar{\mathcal{D}}^2 B_i^\alpha(\hat{T})$. Hence, $\mathcal{O}R(\hat{T}) \sim 4k\, \hat{c}^\mu \xi \sigma_\mu \bar{\xi}$ implies the triviality of ω^3 in the gauge invariant characteristic cohomology ($\omega^3 \approx d\omega^2$ with $\omega^2 = \frac{1}{4} dx^\mu dx^\nu \varepsilon_{\mu\nu\rho\sigma} S^{\rho\sigma}$). The proof of the reversed implication is more involved. Let me just note that one can prove it using the descent equations, by showing that the triviality of ω^3 in the gauge invariant characteristic cohomology implies that the cocycle of $(s_{\rm ext} + d)$ which arises from $\mathcal{O}R(\hat{T})$ (see section 3) is trivial, up to a function (6.35), in the cohomology of $(s_{\rm ext} + d)$ on gauge invariant functions that do not depend on the ghost fields (treating the antifields as gauge covariant quantities).

that these Noether currents are not contravariant Lorentz-vector fields (note that j^μ in (6.36) is a contravariant Lorentz-vector field since $R(\hat{T})$ is Lorentz-invariant owing to $R(\hat{T})\in\mathfrak{T}_{\rm inv}$).

The "generic" representatives of the gauge invariant characteristic cohomology in form-degree 3 involve gauge invariant Noether currents, i.e., they correspond to nontrivial global symmetries of the lagrangian by Noethers first theorem [16]. In addition there are representatives which are trivial in the characteristic cohomology but nevertheless nontrivial in the gauge invariant characteristic cohomology [accordingly the corresponding functions $\mathcal{O}R(\hat{T})$ are nontrivial in $H^3(s_{\rm ext},\mathfrak{T}_{\rm inv})$ but trivial in $H^3(s_{\rm ext},\mathfrak{F})$]. These are exhausted by the 3-forms dB^a and $(dA^{i_A})(\star dB^a)$, as can be shown as analogous results in [15, 7], and do have counterparts in $H^3(s_{\rm susy},\mathfrak{T}_{\rm inv})$: dB^a and $(dA^{i_A})(\star dB^a)$ correspond to functions $R(\hat{T})$ given by ϕ^a and the imaginary part of $\hat{\lambda}^{i_A}\hat{\psi}^a+(1/2)\phi^a\mathcal{D}\hat{\lambda}^{i_A}$, respectively:

$$R(\hat{T}) = -\frac{1}{2}k_a\phi^a \quad\Rightarrow\quad \mathcal{O}R(\hat{T})=k_a\mathcal{H}^a,\qquad j^\mu=k_aH^{a\mu}\,, \tag{6.39}$$

$$\begin{aligned} R(\hat{T}) &= k_{i_A a}(-\mathrm{i}\hat{\lambda}^{i_A}\hat{\psi}^a+\mathrm{i}\bar{\hat{\lambda}}^{i_A}\bar{\hat{\psi}}^a-\mathrm{i}\phi^a\mathcal{D}\hat{\lambda}^{i_A}) \quad\Rightarrow\\ j^\mu &\approx k_{i_A a}\left[-2\varepsilon^{\mu\nu\rho\sigma}F^{i_A}_{\nu\rho}H^a_\sigma+4\partial_\nu(F^{i_A\nu\mu}\phi^a+\lambda^{i_A}\sigma^{\mu\nu}\psi^a+\bar{\lambda}^{i_A}\bar{\sigma}^{\mu\nu}\bar{\psi}^a)\right]. \end{aligned} \tag{6.40}$$

Examples of representatives containing nontrivial Noether currents arise from functions $R(\hat{T})$ that are linear combinations of the real parts of $\hat{\lambda}^{i_{\rm f}}\hat{\psi}^a$:

$$\begin{aligned} R(\hat{T}) &= \frac{1}{2}k_{i_{\rm f}a}(\hat{\lambda}^{i_{\rm f}}\hat{\psi}^a+\bar{\hat{\lambda}}^{i_{\rm f}}\bar{\hat{\psi}}^a) \quad\Rightarrow\\ j^\mu &\approx k_{i_{\rm f}a}\left[2F^{i_{\rm f}\mu\nu}H^a_\nu+\partial_\nu(\varepsilon^{\nu\mu\rho\sigma}F^{i_A}_{\rho\sigma}\phi^a+2\mathrm{i}\lambda^{i_A}\sigma^{\mu\nu}\psi^a-2\mathrm{i}\bar{\lambda}^{i_A}\bar{\sigma}^{\mu\nu}\bar{\psi}^a)\right]. \end{aligned} \tag{6.41}$$

This contains indeed the Noether currents $k_{i_{\rm f}a}F^{i_{\rm f}\mu\nu}H^a_\nu$ of nontrivial global symmetries generated by

$$\Delta A^{i_{\rm f}}_\mu = k_{i_{\rm f}a}H^a_\mu\,,\qquad \Delta B^a_{\mu\nu}=\frac{1}{2}k_{i_{\rm f}a}\,\varepsilon_{\mu\nu\rho\sigma}F^{i_{\rm f}\rho\sigma}\,,\qquad \Delta(\text{other fields})=0\,. \tag{6.42}$$

(c) Almost all representatives of $H(s_{\rm susy},\mathfrak{T}_{\rm inv})$ depend on antifields because the generalized tensor fields $\hat{T}^\tau$ involve antifields, see section 4. Exceptions are the field independent representatives (these are the constants k for $g=0$, the representatives (6.35) for $g=3$, and the representatives $\mathcal{P}\Omega_1+\bar{\mathcal{P}}\bar{\Omega}_2$ with $\Omega_i=k_i\in\mathbb{C}$ for $g=4$), and the representatives $k_a\mathcal{H}^a$, see (6.39) ($\mathcal{H}^a$ does not depend on antifields). In addition there are a few representatives $\mathcal{P}\Omega_1+\bar{\mathcal{P}}\bar{\Omega}_2$ from which the antifield dependence can be removed by subtracting trivial terms, see equation (B.70). The antifield dependence of all other representatives cannot be removed as a consequence of the fact that the commutator algebra of supersymmetry and gauge transformations closes only on-shell. However, this changes when one uses the formulation with the standard auxiliary fields in which the commutator algebra closes off-shell (cf. section 9).

In that formulation,every representative $\mathcal{P}\Omega_1 + \bar{\mathcal{P}}\bar{\Omega}_2$ can be brought in a form that does not depend on antifields. The reason is that, when one uses the auxiliary fields, the algebra (5.9) has an off-shell counterpart which is realized on standard gauge covariant tensor fields (rather than on the generalized tensor fields $\hat{T}^\tau$) and arises from (5.9) by substituting ∇_μ, λ_α^i, $\bar{\lambda}_{\dot\alpha}^i$ and $F_{\mu\nu}^i$ for their hatted counterparts. In the formulation with auxiliary fields, $H^4(s_{\rm susy},\mathfrak{T}_{\rm inv})$ can thus be represented by functions $\mathcal{P}\Omega_1(T) + \bar{\mathcal{P}}\bar{\Omega}_2(T)$ with $\Omega_i(T) = A_i(\varphi,\lambda) + \bar{\mathcal{D}}^2 B_i(T) \in \mathfrak{T}_{\rm inv}$ $(i = 1,2)$ where $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot\alpha}$ are now realized on the ordinary tensor fields T^τ. The antifield dependence of other representatives of $H(s_{\rm susy},\mathfrak{T}_{\rm inv})$ (with $g < 4$) can not be removed by the use of auxiliary fields because of their relation to the characteristic cohomology of the field equations which is not affected by the use of auxiliary fields.

(*d*) The results for $g \geq 3$ in lemma 11 follow from the "QDS-structure" of the supersymmetry multiplets formed by the $\hat{T}^\tau$, see proof of the lemma. These results are thus not restricted to the theories studied here but apply analogously to models with more general lagrangian, or even with different field content, as long as the models have QDS-structure.

6.6 Completion and result of the computation

So far we have determined the part with highest N-degree of the possible representatives of $H(s_{\rm ext},\mathfrak{F})$ by analysing equations (6.17) and (6.18) which involve only $s_{\rm lie}$ and $s_{\rm susy}$. We have found that this part can be assumed to be of the form $f^\Gamma P_\Gamma(\theta,\hat{R})$ where the f^Γ are representatives of $H(s_{\rm susy},\mathfrak{T}_{\rm inv})$ given by lemma 11. To complete the computation of $H(s_{\rm ext},\mathfrak{F})$ we have to examine which functions $f^\Gamma P_\Gamma(\theta,\hat{R})$ can be completed to (inequivalent) solutions of the remaining equations (6.19) through (6.20) involving $s_{\rm curv}$ in addition to $s_{\rm lie}$ and $s_{\rm susy}$. For this purpose the following result is helpful:

Lemma 12 ([17]) *(i) There is no nonvanishing $s_{\rm ext}$-closed function with degree 4 in the $\hat{c}^\mu$.*

(ii) Let $f = f_3 + f_4$ be an $s_{\rm ext}$-cocycle where f_3 and f_4 have degree 3 and 4 in the $\hat{c}^\mu$, respectively. Then f is the $s_{\rm ext}$-variation of a function η_4 with degree 4 in the $\hat{c}^\mu$; η_4 is unique[10] *and the solution of $\delta_-\eta_4 = f_3$ with δ_- as in (B.30).*

Furthermore it is extremely useful to complete the θ_r of the semisimple part of the Yang-Mills group to corresponding "super-Chern-Simons functions" $\hat{q}_r$ given by:

$$\hat{q}_r = m(r)\int_0^1 dt\ \mathrm{tr}_r\left(\hat{C}\mathcal{F}_t^{m(r)-1}\right) \qquad \text{if} \qquad m(r) > 3\,, \tag{6.43}$$

$$\hat{q}_r = \mathrm{tr}_r\left[\hat{C}\mathcal{F}^2 - \frac{1}{2}\hat{C}^3\mathcal{F} + \frac{1}{10}\hat{C}^5 + 3\mathrm{i}\,\Xi\left(\xi\hat{\lambda}\bar{\hat{\lambda}}\bar{\hat{\lambda}} + \bar{\xi}\bar{\hat{\lambda}}\hat{\lambda}\hat{\lambda}\right)\right] \qquad \text{if} \qquad m(r) = 3\,, \tag{6.44}$$

[10]This does not exclude that there are functions g which are not entirely of degree 4 in the $\hat{c}^\mu$ and satisfy $f = s_{\rm ext}g$.

$$\hat{q}_r = \mathrm{tr}_r\left(\hat{C}\mathcal{F} - \frac{1}{3}\hat{C}^3\right) \quad \text{if} \quad m(r) = 2 \quad \text{and} \quad r \leq \mathrm{rank}(\mathfrak{g}_{\mathrm{YM}})\,, \tag{6.45}$$

with $\hat{C}$ and $\{T_i^{(r)}\}$ as in (6.21), and

$$\hat{\lambda}_\alpha = \hat{\lambda}^i_\alpha T_i^{(r)}\,, \qquad \bar{\hat{\lambda}}_{\dot{\alpha}} = \bar{\hat{\lambda}}^i_{\dot{\alpha}} T_i^{(r)}\,, \qquad \mathcal{F} = \mathcal{F}^i T_i^{(r)}\,, \qquad \mathcal{F}_t = t\mathcal{F} + (t^2 - t)\hat{C}^2\,. \tag{6.46}$$

The $\hat{q}$'s satisfy $\hat{q}_r = \theta_r + \cdots$ and the $\hat{q}$'s in (6.43) and (6.44) are s_{ext}-cocycles (in contrast to (6.45)):

Lemma 13. *The super-Chern-Simons functions (6.43) and (6.44) are s_{ext}-closed:*

$$s_{\mathrm{ext}}\hat{q}_r = 0 \qquad \textit{for} \qquad m(r) \geq 3\,. \tag{6.47}$$

Using lemmas 11, 12 and 13 and results on the standard (non-extended) BRST-cohomology one derives the following lemma containing the result of the cohomological analysis:

Lemma 14. *The representatives of $H(s_{\mathrm{ext}}, \mathfrak{F})$ can be grouped into seven types:*

$$s_{\mathrm{ext}}\, f = 0\,, \qquad f \in \mathfrak{F} \qquad \Leftrightarrow \qquad f = \sum_{(i)=(1)}^{(7)} f^{(i)} + s_{\mathrm{ext}}\, h\,, \qquad h \in \mathfrak{F}\,,$$

$$f^{(1)} = P^{(1)}(\hat{q}, \theta_{\mathrm{L}}) \qquad \textit{where} \qquad \frac{\partial P^{(1)}(\hat{q}, \theta_{\mathrm{L}})}{\partial \hat{q}_r} = 0 \qquad \textit{if} \qquad m(r) = 2\,, \tag{6.48}$$

$$f^{(2)} = \left\{\tilde{H}^a + \sum_{r:m(r)=2} X^a_r \frac{\partial}{\partial \hat{q}_r}\right\} P^{(2)}_a(\hat{q}, \theta_{\mathrm{L}})\,, \tag{6.49}$$

$$X^a_r = \left(\frac{1}{8}\phi^a\mathcal{P} - \mathrm{i}\Xi_\mu\bar{\xi}\bar{\sigma}^\mu\hat{\psi}^a + \frac{1}{2}\Xi\hat{\psi}^a\mathcal{D}\right)\mathrm{tr}_r(\hat{\lambda}\hat{\lambda}) - \left(\mathrm{i}\Xi_\mu\xi\bar{\hat{\psi}}^a + \frac{1}{2}\Xi\hat{H}^a_\mu\right)\mathrm{tr}_r(\hat{\lambda}\sigma^\mu\bar{\hat{\lambda}}) + \text{c.c.},$$

$$f^{(3)} = \left\{\tilde{H}^a\hat{R}^b + X^{[ab]} + \sum_{r:m(r)=2} X^{[a}_r\hat{R}^{b]} \frac{\partial}{\partial \hat{q}_r}\right\} P^{(3)}_{[ab]}(\hat{q}, \theta_{\mathrm{L}})\,,$$

$$X^{[ab]} = 2\mathrm{i}\,\Xi_{\mu\nu}\left(\xi\sigma^{\mu\nu}\hat{\psi} - \bar{\xi}\bar{\sigma}^{\mu\nu}\bar{\hat{\psi}}\right)^{[a}\phi^{b]} + \Xi^\mu\left(\mathrm{i}\phi^{[a}\hat{\nabla}_\mu\phi^{b]} - \hat{\psi}^{[a}\sigma_\mu\bar{\hat{\psi}}^{b]}\right), \tag{6.50}$$

$$f^{(4)} = \left\{\tilde{F}^{i_{\mathrm{f}}} + X^{[i_{\mathrm{f}}j_{\mathrm{f}}]}\frac{\partial}{\partial \hat{C}^{j_{\mathrm{f}}}} + X^{i_{\mathrm{f}}a}\frac{\partial}{\partial \hat{R}^a} + \sum_{r:m(r)=2} X^{i_{\mathrm{f}}}_r \frac{\partial}{\partial \hat{q}_r}\right\}\frac{\partial P^{(4)}(\hat{q}, \hat{C}_{\mathrm{free}}, \theta_{\mathrm{L}}, \hat{R})}{\partial \hat{C}^{i_{\mathrm{f}}}}\,,$$

$$X^{[i_{\mathrm{f}}j_{\mathrm{f}}]} = -\mathrm{i}\,\Xi_\mu\hat{\lambda}^{[i_{\mathrm{f}}}\sigma^\mu\bar{\hat{\lambda}}^{j_{\mathrm{f}}]}\,, \qquad X^{i_{\mathrm{f}}a} = \Xi_\mu\hat{\lambda}^{i_{\mathrm{f}}}\sigma^\mu\bar{\xi}\phi^a + \mathrm{i}\,\Xi\,\hat{\lambda}^{i_{\mathrm{f}}}\psi^a - \text{c.c.}\,,$$

$$X^{i_{\mathrm{f}}}_r = \Xi\bar{\xi}\bar{\hat{\lambda}}^{i_{\mathrm{f}}}\,\mathrm{tr}_r(\hat{\lambda}\hat{\lambda}) + 2\Xi\hat{\lambda}^{i_{\mathrm{f}}}\,\mathrm{tr}_r(\hat{\lambda}\bar{\hat{\lambda}}\bar{\xi}) + \text{c.c.}\,, \tag{6.51}$$

$$f^{(5)} = \left\{\tilde{H}^a\tilde{H}^b + \sum_{r:m(r)=2} X^{[ab]}_r \frac{\partial}{\partial \hat{q}_r} + X^{[ab]c}\frac{\partial}{\partial \hat{R}^c} + X^{[ab](cd)}\frac{\partial^2}{\partial \hat{R}^d\partial \hat{R}^c}\right\} P^{(5)}_{[ab]}(\hat{q}, \theta_{\mathrm{L}}, \hat{R})\,,$$

$$
\begin{aligned}
X_r^{[ab]} &= \Big\{ \frac{\mathrm{i}}{8}\phi^{[a}\bar{\hat\psi}^{b]}\bar\xi\mathcal{P} + \frac{1}{2}\Xi_\mu\bar\xi\bar\sigma^\mu\hat\psi^{[a}(\bar{\hat\psi}\bar\xi - \xi\hat\psi)^{b]} + \Xi_\mu\bar\xi\bar\sigma^{\mu\nu}\bar\xi\hat\psi^{[a}(H + \mathrm{i}\hat\nabla\phi)^{b]}_\nu + \\
&\quad + \frac{\mathrm{i}}{4}\Xi\Big[2\hat H^{[a}_\mu\hat\psi^{b]}\sigma^\mu\bar\xi + (\xi\hat\psi - 3\bar{\hat\psi}\bar\xi)^{[a}\hat\psi^{b]}\mathcal{D} + \\
&\qquad + \phi^{[a}(\hat H + \mathrm{i}\hat\nabla\phi)^{b]}_\mu\bar\xi\bar\sigma^\mu\mathcal{D}\Big]\Big\}\,\mathrm{tr}_r(\hat\lambda\hat\lambda) + \\
&\quad + \Big\{\Xi_\mu\xi\hat\psi^{[a}(\xi\hat\psi - \bar{\hat\psi}\bar\xi)^{b]} + 2\mathrm{i}\Xi\xi\hat\psi^{[a}\hat H^{b]}_\mu\Big\}\mathrm{tr}_r(\bar{\hat\lambda}\bar\sigma^\mu\hat\lambda) + \text{c.c.}\,, \\
X^{[ab]c} &= \Xi_{\mu\nu}\Big\{2\xi\sigma^{\mu\nu}\hat\psi^{[a}\xi\hat\psi^{b]}\phi^c + \xi\sigma^\mu\bar{\hat\psi}^{[a}\hat\psi^{b]}\sigma^\nu\bar\xi\phi^c - \\
&\qquad - 2\xi\sigma^{\mu\nu}\xi\phi^{[a}\hat\psi^{b]}\hat\psi^c - 2\xi\sigma^\nu\bar\xi\phi^{[a}\bar{\hat\psi}^{b]}\bar\sigma^\mu\hat\psi^c\Big\} + \\
&\quad + \Xi_\mu\Big\{-2\mathrm{i}\xi\sigma^{\mu\nu}\hat\psi^{[a}\hat H^{b]}_\nu\phi^c + \xi\hat\psi^{[a}\hat\nabla^\mu\phi^{b]}\phi^c + \frac{\mathrm{i}}{2}\xi\sigma^\mu\bar{\mathcal{D}}(\phi^{[a}\hat\psi^{b]}\hat\psi^c) + \\
&\qquad + \frac{\mathrm{i}}{2}\phi^{[a}(\xi\mathcal{D} + \bar\xi\bar{\mathcal{D}})(\bar{\hat\psi}^{b]}\bar\sigma^\mu\hat\psi^c) + \frac{\mathrm{i}}{2}\xi\sigma^\mu\bar{\hat\psi}^{[a}\hat\psi^{b]}\hat\psi^c\Big\} + \\
&\quad + \Xi\Big\{\frac{1}{2}\hat\psi^{[a}\sigma^\mu\bar{\hat\psi}^{b]}\hat H^c_\mu - \hat\psi^c\sigma^\mu\bar{\hat\psi}^{[a}\hat H^{b]}_\mu + \\
&\qquad + \frac{\mathrm{i}}{2}(\hat H^{\mu[a}\phi^{b]}\hat\nabla_\mu\phi^c - \hat H^{\mu c}\phi^{[a}\hat\nabla_\mu\phi^{b]} - \phi^c\hat H^{\mu[a}\hat\nabla_\mu\phi^{b]})\Big\} - \text{c.c.}\,, \\
X^{[ab](cd)} &= 2\mathrm{i}\Xi\left(\bar{\hat\psi}^a\bar\xi\bar{\hat\psi}^b\bar\xi\phi^c\phi^d - \phi^{[a}\bar{\hat\psi}^{b]}\bar\xi\xi\hat\psi^{(c}\phi^{d)}\right) - \text{c.c.}\,,
\end{aligned}
\tag{6.52}
$$

$$
\begin{aligned}
f^{(6)} &= \left\{(\mathcal{O}R^\Gamma(\hat T)) + X^{\Gamma i_A}\frac{\partial}{\partial\hat C^{i_A}} + \sum_{r:m(r)=2}X^\Gamma_r\frac{\partial}{\partial\hat q_r} + X^{\Gamma a}\frac{\partial}{\partial\hat R^a}\right\}P_\Gamma(\hat q,\hat C_{\mathrm{Abel}},\theta_{\mathrm{L}},\hat R), \\
&\quad X^{\Gamma i_A} = \Big\{-2\mathrm{i}\Xi_\mu\hat\lambda^{i_A}\sigma^\mu\bar\xi + \frac{1}{2}\Xi(\mathcal{D}\hat\lambda^{i_A}) + 2\Xi\hat\lambda^{i_A}\mathcal{D} - \text{c.c.}\Big\}R^\Gamma(\hat T)\,, \\
&\quad X^\Gamma_r = 8\Xi R^\Gamma(\hat T)\,\mathrm{tr}_r\left(\xi\lambda\bar{\hat\lambda}\bar\xi\right), \\
&\quad X^{\Gamma a} = 2\mathrm{i}\Xi\left(\bar\psi^a\bar\xi - \xi\psi^a + \phi^a\xi\mathcal{D} + \phi^a\bar\xi\bar{\mathcal{D}}\right)R^\Gamma(\hat T)\,,
\end{aligned}
\tag{6.53}
$$

$$
f^{(7)} = \left\{\left(\mathcal{P}\Omega^\Gamma_1 + \bar{\mathcal{P}}\bar\Omega^\Gamma_2\right) + 8\Xi\left(\bar{\hat\lambda}^{i_A}\bar\xi\Omega^\Gamma_1 + \xi\hat\lambda^{i_A}\bar\Omega^\Gamma_2\right)\frac{\partial}{\partial\hat C^{i_A}}\right\}P_\Gamma\left(\hat q,\hat C_{\mathrm{Abel}},\theta_{\mathrm{L}},\hat R\right), \tag{6.54}
$$

where θ_{L}, $\hat C_{\mathrm{Abel}}$ and $\hat C_{\mathrm{free}}$ denote collectively the Lorentz-θ's (6.23), the abelian $\hat C$'s, and the $\hat C^{i_{\mathrm{f}}}$, respectively, c.c. *denotes complex conjugation when $\{T_i^{(r)}\}$ is antihermitean,*[11] *and other notation is as in lemmas 8 and 11. In particular, $R^\Gamma(\hat T)$, Ω^Γ_1 and $\bar\Omega^\Gamma_2$ are functions as $R(\hat T)$, Ω_1 and $\bar\Omega_2$ in (6.30), respectively, and can be assumed not to be of the trivial form given in (6.31).*

7. Consistent deformations, counterterms and anomalies

We shall now discuss the results for the cohomological groups $H^{0,4}(s_{\mathrm{ext}}|d)$ and $H^{1,4}(s_{\mathrm{ext}}|d)$ because of their relevance to algebraic renormalization and consistent deformations. In algebraic renormalization, $H^{0,4}(s_{\mathrm{ext}}|d)$ yields the possible counterterms that are Poincaré invariant, gauge invariant and $N = 1$ supersymmetric on-shell, and $H^{1,4}(s_{\mathrm{ext}}|d)$ provides the Poincaré invariant candidate gauge and supersymmetry anomalies to lowest nontrivial order, cf. [18, 4, 7].

[11] If $\{T_i^{(r)}\}$ is not antihermitean, then +c.c. denotes the addition of terms that were the complex conjugation if $\{T_i^{(r)}\}$ was antihermitean (i.e., using $-T_i^{(r)}$ in place of $T_i^{(r)\dagger}$).

The other applications concern the Poincaré invariant and $N = 1$ supersymmetric consistent deformations of the classical theories. In this context a deformation is called consistent if it is a continuous deformation[12] of the lagrangian and its gauge symmetries such that the deformed lagrangian is invariant under the deformed gauge transformations up to a total divergence. A deformation is considered trivial when it can be removed through field redefinitions. Such deformations of n-dimensional gauge theories are completely controlled by $H^{0,n}(s|d)$ and $H^{1,n}(s|d)$ where n is the spacetime dimension and s is the standard (non-extended) BRST differential [6, 19]. In particular, $H^{0,n}(s|d)$ provides the nontrivial deformations to first order in the deformation parameters, while $H^{1,n}(s|d)$ yields all possible restrictions or obstructions to the extendability of the deformations to second and all higher orders. Analogously the cohomology groups $H^{0,n}(s_{\rm ext}|d)$ and $H^{1,n}(s_{\rm ext}|d)$ of an extended BRST differential for gauge and global symmetries control those consistent deformations that are invariant under the global symmetry transformations contained in $s_{\rm ext}$, where these transformations may get nontrivially deformed but their commutator algebra does not change on-shell modulo gauge transformations. The latter statement on the algebra of the global symmetries holds because this algebra is encoded in the extended BRST-transformations of the constant ghosts. Representatives of $H^{0,n}(s_{\rm ext}|d)$ do not contain terms that would modify the extended BRST-transformations of the constant ghosts because they do not involve "global antifields" conjugate to the constant ghosts.[13] Hence, $H^{0,n}(s_{\rm ext}|d)$ yields only deformations that preserve the algebra of the global symmetries contained in $s_{\rm ext}$ (in contrast, the algebra of the gauge transformations may get nontrivially deformed). The consistent deformations which arise from our results on $H^{0,4}(s_{\rm ext}|d)$ are thus precisely those which are Poincaré invariant and $N = 1$ supersymmetric, where the supersymmetry transformations may get nontrivially deformed but their algebra is still the standard $N = 1$ supersymmetry algebra (on-shell, modulo gauge transformations). In view of the results derived in [21] it is unlikely that there are more general physically reasonable deformations (which change nontrivially the $N = 1$ supersymmetry algebra) but this question is beyond the scope of this work.

We shall now spell out explicitly the antifield independent parts of the representatives of $H^{0,4}(s_{\rm ext}|d)$. These give the first order deformations of the lagrangian, and the possible counterterms to the lagrangian that are invariant, at least on-shell, under the gauge, Lorentz and supersymmetry transformations up to total divergences (the parts with antifield numbers 1 and 2 of the representatives of $H^{0,4}(s_{\rm ext}|d)$ yield the corresponding first order deformations of, and counterterms to, the symmetry transformations and their commutator algebra, respectively). As explained in section 3, the representatives of $H^{0,4}(s_{\rm ext}|d)$ are obtained from those of $H^4(s_{\rm ext},\mathfrak{F})$ by substituting $c^\mu + dx^\mu$ for c^μ and then picking the 4-form of the resultant expression. The representatives of $H^4(s_{\rm ext},\mathfrak{F})$ are obtained from lemma 14 by selecting from among the functions $f^{(i)}$ those with ghost number 4.[14] Up to

[12] A deformation is called continuous if it is a formal power series in deformation parameters.

[13] See [13] for the concept of such antifields, and [20] for a discussion of deformations of the global symmetry algebra in the framework of the extended antifield formalism.

[14] There are no functions $f^{(1)}$ or $f^{(3)}$ with ghost number 4. The functions $f^{(2)}$ with ghost number 4 arise from polynomials $P_a^{(2)}$ that are linear combinations $k_a^r\hat{q}_r$ of the $\hat{q}_r$ with $m(r) = 2$ and of the Lorentz-θ's

trivial terms, the antifield independent part of the general representative of $H^{0,4}(s_{\rm ext}|d)$ is:

$$
\begin{aligned}
\omega^{0,4}|_{\Phi^*=0} &= d^4x\Big(\overset{(1)}{L}_{\rm CM} + \overset{(1)}{L}_{\rm SYM} + \overset{(1)}{L}_{\rm St} + \overset{(1)}{L}_{\rm FT} + \overset{(1)}{L}_{\rm Noe,CM',CS,FI} + \overset{(1)}{L}_{\rm generic}\Big) \\
\overset{(1)}{L}_{\rm CM} &= \sum_{r:m(r)=2} k_a^r \Big\{ H_\mu^a\, {\rm tr}_r \Big[\varepsilon^{\mu\nu\rho\sigma}\Big(A_\nu\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma\Big) - \lambda\sigma^\mu\bar\lambda\Big] + \\
&\qquad + \phi^a\, {\rm tr}_r\Big[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + {\rm i}\nabla_\mu\lambda\sigma^\mu\bar\lambda - {\rm i}\lambda\sigma^\mu\nabla_\mu\bar\lambda - \frac{1}{4}(\mathcal{D}\lambda)^2\Big] - \\
&\qquad - {\rm tr}_r\Big[(\psi^a\sigma^{\mu\nu}\lambda + \bar\psi^a\bar\sigma^{\mu\nu}\bar\lambda)F_{\mu\nu} - \frac{\rm i}{2}(\psi^a\lambda + \bar\psi^a\bar\lambda)\mathcal{D}\lambda\Big]\Big\} \qquad (7.1) \\
\overset{(1)}{L}_{\rm SYM} &= k_{[i_{\rm f}j_{\rm f}k_{\rm f}]}\Big(-\frac{1}{2}A_\mu^{i_{\rm f}}A_\nu^{j_{\rm f}}F^{k_{\rm f}\mu\nu} + {\rm i}\lambda^{i_{\rm f}}\sigma^\mu\bar\lambda^{j_{\rm f}}A_\mu^{k_{\rm f}}\Big) \qquad (7.2) \\
\overset{(1)}{L}_{\rm St} &= k_{i_{\rm f}a}\Big(\frac{1}{2}F^{i_{\rm f}\mu\nu}B_{\mu\nu}^a + \lambda^{i_{\rm f}}\psi^a + \bar\lambda^{i_{\rm f}}\bar\psi^a\Big) \qquad (7.3) \\
\overset{(1)}{L}_{\rm FT} &= k_{[ab]c}\Big[\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}H_\mu^aH_\nu^bB_{\rho\sigma}^c - H^{\mu a}\phi^b\partial_\mu\phi^c + H^{\mu c}\phi^a\partial_\mu\phi^b + \phi^cH^{\mu a}\partial_\mu\phi^b + \\
&\qquad + {\rm i}\psi^a\sigma^\mu\bar\psi^bH_\mu^c + {\rm i}(\psi^a\sigma^\mu\bar\psi^c - \psi^c\sigma^\mu\bar\psi^a)H_\mu^b\Big] \qquad (7.4) \\
\overset{(1)}{L}_{\rm Noe,CM',CS,FI} &= \Big(\frac{1}{2}A_\mu^{i_A}\bar\sigma^{\mu\alpha\dot\alpha}[\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot\alpha}] - {\rm i}(\mathcal{D}\lambda^{i_A}) - 2{\rm i}\lambda^{i_A}\mathcal{D} + 2{\rm i}\bar\lambda^{i_A}\bar{\mathcal{D}}\Big)R_{i_A}(T) \qquad (7.5) \\
\overset{(1)}{L}_{\rm generic} &= \mathcal{D}^2[A_1(\varphi,\lambda) + \bar{\mathcal{D}}^2B_1(T)] + \bar{\mathcal{D}}^2[\bar A_2(\bar\varphi,\bar\lambda) + \mathcal{D}^2\bar B_2(T)] \qquad (7.6)
\end{aligned}
$$

where the k's are complex numbers, $R_{i_A}(\hat T)$, $A_i(\varphi,\hat\lambda)$ and $B_i(\hat T)$ are functions as in lemma 11, T^τ, $\mathcal{D}_\alpha T^\tau$, $\bar{\mathcal{D}}_{\dot\alpha}T^\tau$ denote the antifield independent parts of $\hat T^\tau$, $\mathcal{D}_\alpha\hat T^\tau$, $\bar{\mathcal{D}}_{\dot\alpha}\hat T^\tau$, respectively, $[\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot\alpha}]$ is the commutator of $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot\alpha}$, and A_μ, λ, $\bar\lambda$ are matrices constructed from the gauge fields and gauginos:

$$
\begin{aligned}
T^\tau &= \hat T^\tau|_{\Phi^*=0}\,, & \mathcal{D}_\alpha T^\tau &= (\mathcal{D}_\alpha\hat T^\tau)|_{\Phi^*=0}\,, \\
\bar{\mathcal{D}}_{\dot\alpha}T^\tau &= (\bar{\mathcal{D}}_{\dot\alpha}\hat T^\tau)|_{\Phi^*=0}\,, & [\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot\alpha}] &= \mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\alpha} - \bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\alpha\,, \\
A_\mu = A_\mu^iT_i^{(r)}\,, \qquad \lambda_\alpha &= \lambda_\alpha^iT_i^{(r)}\,, & \bar\lambda_{\dot\alpha} &= \bar\lambda_{\dot\alpha}^iT_i^{(r)}\,. \qquad (7.7)
\end{aligned}
$$

Analogously one obtains the representatives of $H^{1,4}(s_{\rm ext}|d)$ from the functions $f^{(i)}$ with ghost number 5 given in lemma 14. The antifield independent parts of these representatives are (the superscripts indicate from which $f^{(i)}$ they derive):

$$
\begin{aligned}
\mathcal{A}^{(1)} = \sum_{r:m(r)=3} k^r\, {\rm tr}_r\Big\{&Cd\Big(AdA + \frac{1}{2}A^3\Big) + {\rm i}(\xi\sigma\bar\lambda + \lambda\,\sigma\bar\xi)\Big(AdA + (dA)A + \frac{3}{2}A^3\Big) + \\
&+ 3{\rm i}d^4x\,\big(\bar\xi\bar\lambda\lambda\lambda + \xi\lambda\bar\lambda\bar\lambda\big)\Big\} \qquad (7.8)
\end{aligned}
$$

where, however, only $k_a^r\hat q_r$ yields deformations and counterterms of the lagrangian, symmetry transformations and their algebra (the representatives with Lorentz-θ's have antifield number 3). The functions $f^{(4)}$, $f^{(5)}$, $f^{(6)}$, and $f^{(7)}$ with ghost number 4 arise from $P^{(4)} = (1/6)k_{[i_{\rm f}j_{\rm f}k_{\rm f}]}\hat C^{i_{\rm f}}\hat C^{j_{\rm f}}\hat C^{k_{\rm f}} - {\rm i}k_{i_{\rm f}a}\hat C^{i_{\rm f}}\hat R^a$, $P^{(5)}_{[ab]} = {\rm i}k_{[ab]c}\hat R^c$, $P_\Gamma = -{\rm i}\hat C^{i_A}$, and $P_\Gamma = 1$ respectively, where we introduced cosmetic factors $1/6$ and $\pm{\rm i}$.

$$\mathcal{A}^{(4a)} = k_{[i_\mathrm{f}j_\mathrm{f}k_\mathrm{f}l_\mathrm{f}]}\left\{\frac{1}{2}(\star F^{i_\mathrm{f}})A^{j_\mathrm{f}}A^{k_\mathrm{f}}C^{l_\mathrm{f}} + \frac{1}{6}(\xi\sigma\bar\lambda^{i_\mathrm{f}} - \lambda^{i_\mathrm{f}}\sigma\bar\xi)A^{j_\mathrm{f}}A^{k_\mathrm{f}}A^{l_\mathrm{f}} + \mathrm{i}d^4x\lambda^{i_\mathrm{f}}\sigma^\mu\bar\lambda^{j_\mathrm{f}}A^{k_\mathrm{f}}_\mu C^{l_\mathrm{f}}\right\} \tag{7.9}$$

$$\begin{aligned}\mathcal{A}^{(4b)} = \sum_{r:m(r)=2} k^r_{i_\mathrm{f}}\Big\{&(\star F^{i_\mathrm{f}})\,\mathrm{tr}_r(CdA + \mathrm{i}A\xi\sigma\bar\lambda + \mathrm{i}A\lambda\sigma\bar\xi) + \\ &+ (\xi\sigma\bar\lambda^{i_\mathrm{f}} - \lambda^{i_\mathrm{f}}\sigma\bar\xi)\,\mathrm{tr}_r\left(AdA + \frac{2}{3}A^3\right) + \\ &+ d^4x\left[\bar\xi\bar\lambda^{i_\mathrm{f}}\,\mathrm{tr}_r(\lambda\lambda) + 2\lambda^{i_\mathrm{f}}\,\mathrm{tr}_r(\lambda\bar\lambda\bar\xi) + \mathrm{c.c.}\right]\Big\}\end{aligned} \tag{7.10}$$

$$\begin{aligned}\mathcal{A}^{(4c)} = k_{[i_\mathrm{f}j_\mathrm{f}]a}\Big\{&(\star F^{i_\mathrm{f}})(A^{j_\mathrm{f}}Q^a + C^{j_\mathrm{f}}B^a) + (\xi\sigma\bar\lambda^{i_\mathrm{f}} - \lambda^{i_\mathrm{f}}\sigma\bar\xi)A^{j_\mathrm{f}}B^a + \\ &+ d^4x\left(\frac{\mathrm{i}}{2}\lambda^{i_\mathrm{f}}\sigma^\mu\bar\lambda^{j_\mathrm{f}}Q^a_\mu + \mathrm{i}\lambda^{i_\mathrm{f}}\sigma^\mu\bar\xi A^{j_\mathrm{f}}_\mu - \lambda^{i_\mathrm{f}}\psi^a C^{j_\mathrm{f}} + \mathrm{c.c.}\right)\Big\}\end{aligned} \tag{7.11}$$

$$\begin{aligned}\mathcal{A}^{(5)} = \sum_{r:m(r)=2} k^r_{[ab]}\Big\{&(\star dB^a)(\star dB^b)\,\mathrm{tr}_r(CdA + \mathrm{i}A\xi\sigma\bar\lambda + \mathrm{i}A\lambda\sigma\bar\xi) + \\ &+ 2\mathrm{i}(\star dB^a)(\bar\psi^b\bar\xi - \xi\psi^b)\,\mathrm{tr}_r\left(AdA + \frac{2}{3}A^3\right) + \\ &+ d^4x\left[2\mathrm{i}\bar\psi^a\bar\xi H^b_\mu\,\mathrm{tr}_r(\lambda\sigma^\mu\bar\lambda) + \mathcal{P}^{ab}\,\mathrm{tr}_r(\lambda\lambda) + \mathrm{c.c.}\right]\Big\} \\ \text{with}\qquad \mathcal{P}^{ab} = &\ \frac{\mathrm{i}}{8}\phi^a\bar\psi^b\bar\xi\mathcal{D}^2 + \frac{\mathrm{i}}{4}\phi^a(H_\mu + \mathrm{i}\partial_\mu\phi)^b\bar\xi\bar\sigma^\mu\mathcal{D} + \\ &+ \frac{\mathrm{i}}{4}(\xi\psi^a - 3\bar\psi^a\bar\xi)\psi^b\mathcal{D} + \frac{\mathrm{i}}{2}H^a_\mu\psi^b\sigma^\mu\bar\xi\end{aligned} \tag{7.12}$$

$$\begin{aligned}\mathcal{A}^{(6a)} = d^4x\Big\{&\frac{1}{4}C^{i_A}A^{j_A}_{\alpha\dot\alpha}\,[\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot\alpha}] + 2\mathrm{i}A^{i_A}_\mu A^{j_A}_\nu\,\xi\sigma^{\mu\nu}\mathcal{D} - 2\mathrm{i}C^{i_A}\lambda^{j_A}\mathcal{D} - \\ &- \frac{\mathrm{i}}{2}C^{i_A}(\mathcal{D}\lambda^{j_A}) + 2\xi\sigma^\mu\bar\lambda^{i_A}A^{j_A}_\mu + \mathrm{c.c.}\Big\}R_{[i_Aj_A]}(T)\end{aligned} \tag{7.13}$$

$$\mathcal{A}^{(6b)} = d^4x\left\{\frac{1}{4}Q^a_{\alpha\dot\alpha}\,[\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot\alpha}] + 4\mathrm{i}B^a_{\mu\nu}\xi\sigma^{\mu\nu}\mathcal{D} - 2\mathrm{i}\xi\psi^a + 2\mathrm{i}\phi^a\xi\mathcal{D} + \mathrm{c.c.}\right\}R_a(T) \tag{7.14}$$

$$\begin{aligned}\mathcal{A}^{(7)} = &\ d^4x\left(C^{i_A}\mathcal{D}^2 - 4\mathrm{i}A^{i_A}_\mu\,\bar\xi\bar\sigma^\mu\mathcal{D} + 8\bar\lambda^{i_A}\bar\xi\right)\left[A_{i_A}(\varphi,\lambda) + \bar{\mathcal{D}}^2B_{i_A}(T)\right] + \\ &+ d^4x\left(C^{i_A}\bar{\mathcal{D}}^2 + 4\mathrm{i}A^{i_A}_\mu\,\xi\sigma^\mu\bar{\mathcal{D}} + 8\xi\lambda^{i_A}\right)\left[A'_{i_A}(\bar\varphi,\bar\lambda) + \mathcal{D}^2B'_{i_A}(T)\right]\end{aligned} \tag{7.15}$$

where $R_{[i_Aj_A]}(\hat T)$ and $R_a(\hat T)$ are functions as $R(\hat T)$ in lemma 11, $A_{i_A}(\varphi,\lambda)$, $B_{i_A}(T)$, $A'_{i_A}(\bar\varphi,\bar\lambda)$, $B'_{i_A}(T)$ are gauge invariant and Lorentz invariant functions, c.c. is used as in lemma 14, and

$$C = C^iT^{(r)}_i\,,\quad Q^a = dx^\mu Q^a_\mu\,,\quad \sigma_{\alpha\dot\alpha} = dx^\mu\sigma_{\mu\,\alpha\dot\alpha}\,,\quad A^i = dx^\mu A^i_\mu\,,\quad A = A^iT^{(r)}_i\,, \tag{7.16}$$

$$\star F^i = \tfrac{1}{4}dx^\mu dx^\nu\varepsilon_{\mu\nu\rho\sigma}F^{i\rho\sigma}\,,\quad B^a = \tfrac{1}{2}dx^\mu dx^\nu B^a_{\mu\nu}\,,\quad \star dB^a = dx^\mu H^a_\mu\,. \tag{7.17}$$

We leave it to the reader to work out the antifield dependent terms of the representatives of $H^{0,4}(s_\mathrm{ext}|d)$ and $H^{1,4}(s_\mathrm{ext}|d)$ and add the following comments:

(a) $\overset{(1)}{L}_{\mathrm{CM}}$ contains "Chapline-Manton" couplings between the 2-form gauge potentials and non-abelian Chern-Simons 3-forms of the type frequently encountered in supergravity models, see, e.g., [22, 23, 24, 25].

(b) $\overset{(1)}{L}_{\rm SYM}$ contains the cubic interaction vertices of standard super-Yang-Mills theories. In particular it gives rise to deformations of free supersymmetric gauge theories to standard non-abelian super-Yang-Mills theories, with the coefficients $k_{[i_{\rm f}j_{\rm f}k_{\rm f}]}$ becoming the structure constants of the non-abelian gauge group (the Jacobi identity for the structure constants arises at second order of the deformation, cf. [26]; see also comment (h) below).

(c) $\overset{(1)}{L}_{\rm St}$ gives rise to the supersymmetric version of the Stueckelberg mechanism [27] for 2-form gauge potentials.

(d) $\overset{(1)}{L}_{\rm FT}$ contains the trilinear vertices $\varepsilon^{\mu\nu\rho\sigma}H^a_\mu H^b_\nu B^c_{\rho\sigma}$ of "Freedman-Townsend models" [28, 29]. In particular it yields deformations of gauge theories for free linear multiplets to supersymmetric Freedman-Townsend models as derived in [30, 31, 32].

(e) One may distinguish four different types of functions $\overset{(1)}{L}_{\rm Noe,CM',CS,FI}$, depending on the functions $R_{i_A}(T)$ they involve. The simplest choice is $R_{i_A}(T) = k_{i_A} \in \mathbb{C}$. It yields Fayet-Iliopoulos terms [33]:[15]

$$\overset{(1)}{L}_{\rm FI} = -\mathrm{i}k_{i_A}\mathcal{D}\lambda^{i_A}\,. \tag{7.18}$$

All other choices lead to terms containing couplings $A^{i_A}_\mu j^\mu_{i_A}$ of abelian gauge fields to conserved currents $j^\mu_{i_A} = (1/2)\bar\sigma^\mu_{\alpha\dot\alpha}[\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot\alpha}]R_{i_A}(T)$, cf. comment (b) in section 6.5. Generically these conserved currents are nontrivial Noether currents (up to trivial currents). The only exceptions are the currents which arise from $R_{i_A}(T) = -\frac{1}{2}k_{i_A a}\phi^a + k_{i_A j_A a}(-\mathrm{i}\lambda^{j_A}\psi^a + \mathrm{i}\bar\lambda^{j_A}\bar\psi^a - \mathrm{i}\phi^a\mathcal{D}\lambda^{j_A})$, see (6.39) and the text before that equation. These R_{i_A}'s yield Chern-Simons type couplings between abelian gauge fields and the 2-form gauge potentials, and Chapline-Manton couplings of the 2-form gauge potentials to abelian Chern-Simons 3-forms:

$$\overset{(1)}{L}_{\rm CS} = k_{i_A a}A^{i_A}_\mu H^{a\mu} + \cdots = \frac{1}{2}k_{i_A a}\varepsilon^{\mu\nu\rho\sigma}A^{i_A}_\mu \partial_\nu B^a_{\rho\sigma} + \cdots \tag{7.19}$$

$$\overset{(1)}{L}_{\rm CM'} = -2k_{i_A j_A a}\varepsilon^{\mu\nu\rho\sigma}A^{i_A}_\mu F^{j_A}_{\nu\rho}H^a_\sigma + \cdots \tag{7.20}$$

An example for a deformation involving a nontrivial Noether current arises from $R_{i_A}(T) = \frac{1}{2}k_{i_A i_{\rm f} a}(\lambda^{i_{\rm f}}\psi^a + \bar\lambda^{i_{\rm f}}\bar\psi^a)$ which yields

$$\overset{(1)}{L}_{\rm Noe} = 2k_{i_A i_{\rm f} a}A^{i_A}_\mu F^{i_{\rm f}\mu\nu}H^a_\nu + \cdots \tag{7.21}$$

Supersymmetric models with these interaction vertices were constructed in [32].

(f) The invariants (7.6) have been termed "generic" because there are infinitely many of them, with arbitrarily high mass dimensions. For instance, an invariant with mass dimension 8 arises from a contribution $\mathrm{tr}_r(\lambda\lambda\bar\lambda\bar\lambda)$ to B_1,

$$\mathcal{D}^2\bar{\mathcal{D}}^2\,\mathrm{tr}_r(\lambda\lambda\bar\lambda\bar\lambda) = \mathrm{tr}_r(16F_{\mu\nu}F_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} - 4F_{\mu\nu}F_{\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + \cdots)\,. \tag{7.22}$$

[15] In the formulation with auxiliary fields $\mathcal{D}\lambda^{i_A}$ is proportional to the auxiliary field D^{i_A}, cf. section 9.

In the formulation with the standard auxiliary fields (see section 9), all invariants (7.6) can be written as off-shell invariants, cf. comment (c) in section 6.5. Hence, in that formulation deformations (7.6) preserve the form of the supersymmetry and gauge transformations. Notice also that they can be written as standard superspace integrals in a superfield formulation ($\mathcal{D}^2$ and $\bar{\mathcal{D}}^2$ then turn into superspace integrals $\int d^2\theta$ and $\int d^2\bar{\theta}$).

(g) Eq. (7.8) is a supersymmetric generalization of the non-abelian chiral anomalies. Remarkably it is not accompanied by antifield dependent terms, i.e., it is already a complete representative of $H^{1,4}(s_{\mathrm{ext}}|d)$, whether or not one uses auxiliary fields (alternative forms and discussions of supersymmetric non-abelian chiral anomalies can be found in [34]–[52][2][53]–[55]). I note that (7.8) has a direct generalization in supergravity [17, 12].

(h) Eq. (7.9), (7.10) and (7.11) are present only when the Yang-Mills gauge group contains abelian gauge symmetries under which all matter fields are uncharged. They are unlikely to represent anomalies of quantum theories but give important restrictions to consistent deformations at higher orders in the deformation parameters. In particular (7.9) enforces at second order that the coefficients $k_{[i_{\mathrm{f}}j_{\mathrm{f}}k_{\mathrm{f}}]}$ in (7.2) satisfy the Jacobi identity for structure constants of a Lie algebra, cf. [26].

(i) Eq. (7.12) may be viewed as an analogue with ghost number 1 of non-abelian Chapline-Manton type couplings (7.1).

(j) Eq. (7.13) is the analogue with ghost number 1 of the couplings (7.5). Owing to the antisymmetry of $R_{[i_A j_A]}(T)$ these representatives exist only if the Yang-Mills gauge group contains at least two abelian factors. As in the case of the couplings (7.5) one may distinguish between different types of representatives (7.13), depending on the choice of $R_{[i_A j_A]}(T)$. The simplest choice is $R_{[i_A j_A]}(T) = k_{[i_A j_A]} \in \mathbb{C}$ and gives the analogue with ghost number 1 of the Fayet-Iliopoulos terms (7.18):

$$k_{[i_A j_A]}\left[-\mathrm{i}C^{i_A}\mathcal{D}\lambda^{j_A} + 2(\xi\sigma^\mu\bar{\lambda}^{i_A} + \lambda^{i_A}\sigma^\mu\bar{\xi})A_\mu^{j_A}\right] d^4x\,. \tag{7.23}$$

All other choices yield representatives containing conserved currents given by (6.39), (6.40) or Noether currents such as in (6.41). The representatives arising from (6.39) do not contain antifields and are somewhat reminiscent of chiral anomalies because they read

$$k_{[i_A j_A]a}C^{i_A}A^{j_A}dB^a + \cdots\,. \tag{7.24}$$

The representatives arising from (6.40) may be viewed as an analogue with ghost number 1 of abelian Chapline-Manton type couplings (7.20). They read:

$$k_{[i_A j_A]k_A a}C^{i_A}A^{j_A}(dA^{k_A})(\star dB^a) + \cdots\,. \tag{7.25}$$

The representatives containing the currents (6.41) read

$$k_{[i_A j_A]i_{\mathrm{f}}a}C^{i_A}A^{j_A}(\star dA^{i_{\mathrm{f}}})(\star dB^a) + \cdots\,. \tag{7.26}$$

(k) Similarly there are several types of representatives (7.14). The simplest arise from $R_a(T) = k_a \in \mathbb{C}$ and read

$$2\mathrm{i}k_a(\bar{\psi}^a\bar{\xi} - \xi\psi^a)d^4x\,. \tag{7.27}$$

The other representatives (7.14) involve conserved currents and are of the form

$$d^4x\,(Q^a_\mu j^\mu_a + \cdots)\,, \qquad j^\mu_a = \frac{1}{2}\sigma^\mu_{\alpha\dot{\alpha}}[\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}]R_a(T). \tag{7.28}$$

Again, one may distinguish between representatives containing the currents (6.39), (6.40), or Noether currents such as in (6.41). Those with the currents (6.39) are not accompanied by antifields and explicitly given by (one can assume $k_{ab} = k_{[ab]}$ because the part with $k_{(ab)}$ is trivial):

$$k_{[ab]}\left[Q^a dB^b - \mathrm{i}B^a_{\mu\nu}(\xi\sigma^{\mu\nu}\psi^b + \bar{\xi}\bar{\sigma}^{\mu\nu}\bar{\psi}^b)d^4x + 2\mathrm{i}\phi^a(\xi\psi^b - \bar{\psi}^b\bar{\xi})d^4x\right]. \tag{7.29}$$

(l) The representatives (7.15) are the counterparts with ghost number 1 of the invariants (7.6). The simplest nontrivial representatives (7.15) arise from $A_{i_A} = k_{i_A} \in \mathbb{C}$, $A'_{i_A} = k'_{i_A} \in \mathbb{C}$, $B_{i_A} = B'_{i_A} = 0$. They read

$$8(k_{i_A}\xi\lambda^{i_A} + k'_{i_A}\bar{\lambda}^{i_A}\bar{\xi})\,d^4x\,. \tag{7.30}$$

Significant representatives (7.15) are in particular the supersymmetric generalizations of abelian chiral anomalies. They arise from $A_{i_A}(\varphi,\lambda) = -\mathrm{i}k^r_{i_A}\,\mathrm{tr}_r(\lambda\lambda) - \mathrm{i}k_{(i_A j_A k_A)}\lambda^{j_A}\lambda^{k_A}$, $A'_{i_A}(\bar{\varphi},\bar{\lambda}) = \mathrm{i}k^r_{i_A}\,\mathrm{tr}_r(\bar{\lambda}\bar{\lambda}) + \mathrm{i}k_{(i_A j_A k_A)}\bar{\lambda}^{j_A}\bar{\lambda}^{k_A}$, $B_{i_A} = B'_{i_A} = 0$. This yields supersymmetrized abelian chiral anomalies in a form as in eq. (5.14) of [1]. By adding a coboundary of $H^{1,4}(s_{\mathrm{ext}}|d)$ to these anomalies, they can be brought to a form analogous to (7.8) as can be inferred from (B.70).

(m) Notice: when the Yang-Mills gauge group is semisimple and no linear multiplets are present, (7.6) and (7.8) exhaust the nontrivial representatives of $H^{0,4}(s_{\mathrm{ext}}|d)$ and $H^{1,4}(s_{\mathrm{ext}}|d)$, respectively. Indeed, all other representatives require that the Yang-Mills gauge group contains abelian factors or that linear multiplets are present. Recall that the representatives (7.6) can be written as off-shell invariants when one uses the auxiliary fields, see item (f). Hence, these off-shell invariants provide all Lorentz-invariant and $N = 1$ supersymmetric deformations of standard super-Yang-Mills theories with semisimple gauge group (to all orders!).

8. Remarks on the cohomology in negative ghost numbers

In [5] it was shown that the standard (non-extended) BRST cohomological groups $H^{p-n,n}(s|d)$ in n-dimensional theories for negative ghost numbers are isomorphic to the characteristic cohomology in form-degree p $(0 < p < n)$ whose representatives are conserved local p-forms ($d\omega^p \approx 0$). The cohomological groups $H^{p-4,4}(s_{\mathrm{ext}}|d)$, $0 < p < 4$, have a similar interpretation: they correspond to conserved local p-forms that are invariant under $N = 1$ supersymmetry and Poincaré transformations up to trivially conserved p-forms.

To show this we use that each representative of $H^{p-4,4}(s_{\rm ext}|d)$ is related through the descent equations for $s_{\rm ext}$ and d to a local p-form $\omega^{0,p}$ with ghost number 0 which satisfies $s_{\rm ext}\omega^{-1,p+1}+d\omega^{0,p}=0$ and $s_{\rm ext}\omega^{0,p}+d\omega^{1,p-1}=0$ (see proof of lemma 1). The parts of these equations with antifield number 0 read (see section 6.1 for the notation):

$$\delta\omega_1^{-1,p+1}+d\omega_0^{0,p}=0 \quad\Leftrightarrow\quad d\omega_0^{0,p}\approx 0 \tag{8.1}$$

$$\gamma_{\rm ext}\omega_0^{0,p}+\delta\omega_1^{0,p}+d\omega_0^{1,p-1}=0 \quad\Leftrightarrow\quad \gamma_{\rm ext}\omega_0^{0,p}\approx -d\omega_0^{1,p-1}, \tag{8.2}$$

where subscripts denote the antifield numbers ($\omega_k^{g,p}$ is the part with antifield number k contained in $\omega^{g,p}$). Notice that $\omega_0^{0,p}$ has vanishing ghost and antifield number and thus does not depend on ghost fields, constant ghosts or antifields. (8.1) shows that it is conserved. Furthermore it is gauge invariant, $N=1$ supersymmetric and Poincaré invariant up to trivially conserved p-forms. This is seen from (8.2) because $\gamma_{\rm ext}$ contains the gauge transformations (with ghost fields in place of gauge parameters), the $N=1$ supersymmetry transformations and the Poincaré transformations (multiplied by the corresponding constant ghosts, respectively).

Moreover $\omega_0^{0,p}$ is trivial in the characteristic cohomology ($\omega_0^{0,p}\approx d\omega_0^{0,p-1}$) iff the corresponding representative of $H^{p-4,4}(s_{\rm ext}|d)$ is trivial in the cohomology of $s_{\rm ext}$ modulo d. For $p<3$ we had seen this already in section 6.1 where we found that $H^{-3,4}(s_{\rm ext}|d)$ and $H^{-2,4}(s_{\rm ext}|d)$ are isomorphic to the characteristic cohomology in form-degrees 1 and 2, respectively. For $p=3$ the assertion can be proved by the arguments used in the proof of lemma 7: one finds that the part $\omega_1^{-1,4}$ of a nontrivial representative of $H^{-1,4}(s_{\rm ext}|d)$ is a nontrivial representative of $H_1^4(\delta|d)$.[16] As shown in [5], it thus corresponds to a nontrivial global symmetry and, via descent equations for δ and d, to a nontrivial conserved 3-form containing the Noether current of this global symmetry.[17] Hence, all nontrivial representatives of $H^{-1,4}(s_{\rm ext}|d)$ correspond to nontrivial global symmetries of the action and nontrivial conserved currents. These currents are $N=1$ supersymmetric and Poincaré invariant up to trivial conserved currents (see discussion above). Accordingly the global symmetries corresponding to representatives of $H^{-1,4}(s_{\rm ext}|d)$ commute on-shell with the $N=1$ supersymmetry transformations and the Poincaré transformations up to gauge transformations.[18]

$H^{-3,4}(s_{\rm ext}|d)$ and $H^{-2,4}(s_{\rm ext}|d)$ were already given in lemma 7. $H^{-1,4}(s_{\rm ext}|d)$ can be obtained from $H^3(s_{\rm ext},\mathfrak{F})$ which has three types of representatives as one infers from lemma 14: $f^{(3)}$ with $P^{(3)}_{[ab]}=k_{[ab]}\in\mathbb{C}$ which correspond to the global symmetries of the action under rotations acting on the indices a of the linear multiplets, $f^{(4)}$ with

[16] Applying the arguments in the proof of lemma 7 to the case $g=-1$ one finds that the nontrivial representatives have $\underline{k}=1$ (but not $\underline{k}=2$ or $\underline{k}=3$ because there are no Lorentz-invariant homogeneous polynomials of degree 1 or 2 in the constant ghosts).

[17] In this context a global symmetry is called trivial if it equals a gauge transformation (in general with field dependent parameters) on-shell. A conserved current j^μ is called trivial if $j^\mu\approx\partial_\nu S^{\nu\mu}$ for some local functions $S^{\nu\mu}=-S^{\mu\nu}$.

[18] Let Δ_A be (infinitesimal) global symmetry transformations and j^μ_A the corresponding conserved currents. One has $\Delta_A j^\mu_B\sim j^\mu_{[A,B]}$ where $j^\mu_{[A,B]}$ is the Noether current of $[\Delta_A,\Delta_B]$ and $\sim$ is equality up to trivial currents [56, 57]. Furthermore a current is trivial iff the corresponding global symmetry is trivial [5].

$P^{(4)} = (1/2)k_{i_f j_f}\hat{C}^{i_f}\hat{C}^{j_f}$ ($k_{i_f j_f} \in \mathbb{C}$) which correspond to the global symmetries under rotations of the indices i_f, and $f^{(6)} = \mathcal{O}R(\hat{T})$ where, however, (6.35), (6.39) and (6.40) do not yield nontrivial representatives of $H^{-1,4}(s_{\rm ext}|d)$ ((6.35) does not correspond to any representatives of $H^{-1,4}(s_{\rm ext}|d)$ because it is field independent, while (6.39) and (6.40) are trivial in $H^3(s_{\rm ext},\mathfrak{F})$ though nontrivial in $H^3(s_{\rm ext},\mathfrak{T}_{\rm inv})$).

Comments.

(*a*) We just observed that the representatives of $H^{-1,4}(s_{\rm ext}|d)$ correspond to nontrivial conserved currents that are $N=1$ supersymmetric and Poincaré invariant up to trivially conserved currents, and to the corresponding global symmetries which commute on-shell with the $N=1$ supersymmetry transformations and the Poincaré transformations up to gauge transformations. However, the correspondence is not one-to-one because there are conserved currents and global symmetries of this type which have no counterpart in $H^{-1,4}(s_{\rm ext}|d)$. Indeed, consider the currents $j^{\mu a} = \partial^\mu \phi^a$. They are the Noether currents of the global symmetries under constant shifts of the ϕ^a. These shift symmetries are evidently nontrivial. Furthermore they commute with all $N=1$ supersymmetry and Poincaré transformations. Accordingly their currents are conserved, nontrivial, and both $N=1$ supersymmetric and Poincaré invariant up to trivial currents, respectively. Nevertheless they do not give rise to representatives of $H^{-1,4}(s_{\rm ext}|d)$. This can be seen in various ways. One way is the following: the corresponding representatives of $H^3(s_{\rm ext},\mathfrak{F})$ would have dimension -1 and be linear in the fields of the linear multiplets (this follows from the structure of the shift symmetries and their currents), and would thus inevitably have to be proportional to $\mathcal{O}\phi^a$; however, $\mathcal{O}\phi^a$ is trivial in $H^3(s_{\rm ext},\mathfrak{F})$ for one has $\mathcal{O}\phi^a = -2\mathcal{H}^a = -2s_{\rm ext}\hat{R}^a$. Another way is based on the algebraic structure which is associated to the local BRST cohomology and described in section 3 of [13]. In the present case this algebra links the representatives of $H^{-1,4}(s|d)$ corresponding to super-Poincaré symmetries and the symmetries under constant shifts of the ϕ's to the representatives $R^*_a d^4x$ of $H^{-3,4}(s|d)$ in a nontrivial way (owing to the presence of ϕ-dependent gauge transformations in the commutator of two supersymmetry transformations on $B_{\mu\nu}$) which obstructs the existence of representatives of $H^{-1,4}(s_{\rm ext}|d)$ corresponding to the shift symmetries.

(*b*) The p-forms $\omega^{0,p}$ mentioned above, which are related through the descent equations to the representatives of $H^{p-4,4}(s_{\rm ext}|d)$ for $0<p<4$, make up a complete set of field dependent representatives of $H^{0,p}(s_{\rm ext}|d)$ for $p<4$. This can be inferred from comment c) in section 3 using the result that $H^0(s_{\rm ext},\mathfrak{F})$ is represented just by a number (the latter result follows from lemma 14, a number being a solution $f^{(1)}$).

9. Other formulations of supersymmetry

In this section it is shown that our cohomological results do not depend on the chosen formulation of supersymmetry. Alternative well-known formulations with auxiliary fields or linearly realized supersymmetry (in particular, the standard superspace formulation) lead to exactly the same results. The reason is that the additional fields and antifields which occur in these formulations give only rise to trivial pairs and thus do not contribute nontrivially to the cohomology, cf. section 6.2.

9.1 Formulation with auxiliary fields

Let us first discuss the formulation with the standard auxiliary fields so that the commutator algebra of the Poincaré, supersymmetry and gauge transformations closes off-shell. These auxiliary fields are real fields D^i for the super-Yang-Mills multiplets and complex fields F^s for the chiral multiplets. The formulation with these fields differs from the one used here through the following changes as compared to section 2:

1. Field content: one adds the fields D^i, F^s, $\bar{F}_s$ and their antifields D^*_i, F^*_s, $\bar{F}^{*s}$.

2. Lagrangian:

$$\begin{aligned}L = &-\frac{1}{4}\delta_{ij}F^i_{\mu\nu}F^{j\mu\nu} + \frac{\mathrm{i}}{2}\delta_{ij}(\nabla_\mu\bar\lambda^i\bar\sigma^\mu\lambda^j - \bar\lambda^i\bar\sigma^\mu\nabla_\mu\lambda^j) + \frac{1}{2}\delta_{ij}D^iD^j + \\ &+\frac{1}{2}\partial_\mu\phi^t\partial^\mu\phi - \frac{1}{2}H^t_\mu H^\mu + \frac{\mathrm{i}}{2}(\partial_\mu\bar\psi^t\bar\sigma^\mu\psi - \bar\psi^t\bar\sigma^\mu\partial_\mu\psi) + \\ &+\nabla_\mu\bar\varphi\nabla^\mu\varphi + \frac{\mathrm{i}}{4}(\nabla_\mu\bar\chi\bar\sigma^\mu\chi - \bar\chi\bar\sigma^\mu\nabla_\mu\chi) + \frac{1}{4}\bar{F}F + \\ &+\mathrm{i}D^i\bar\varphi T_i\varphi + \bar\varphi T_i\chi\lambda^i - \bar\lambda^i\bar\chi T_i\varphi\,. \end{aligned} \tag{9.1}$$

3. Extended BRST transformations of the fields: the only changes are in the transformations of λ_α, χ_α and their complex conjugates, and the addition of the extended BRST-transformations of the auxiliary fields:

$$\begin{aligned} s_{\mathrm{ext}}\lambda^i_\alpha &= -\,f_{jk}{}^iC^j\lambda^k_\alpha + \hat{c}^\mu\partial_\mu\lambda^i_\alpha - \frac{1}{2}c^{\mu\nu}(\sigma_{\mu\nu}\lambda^i)_\alpha - \mathrm{i}\xi_\alpha D^i + (\sigma^{\mu\nu}\xi)_\alpha F^i_{\mu\nu} \\ s_{\mathrm{ext}}D^i &= -\,f_{jk}{}^iC^jD^k + \hat{c}^\mu\partial_\mu D^i + \xi\sigma^\mu\nabla_\mu\bar\lambda^i + \nabla_\mu\lambda^i\sigma^\mu\bar\xi \\ s_{\mathrm{ext}}\chi_\alpha &= -\,C^iT_i\chi_\alpha + \hat{c}^\mu\partial_\mu\chi_\alpha - \frac{1}{2}c^{\mu\nu}(\sigma_{\mu\nu}\chi)_\alpha + \xi_\alpha F - 2\mathrm{i}(\sigma^\mu\bar\xi)_\alpha\nabla_\mu\varphi \\ s_{\mathrm{ext}}F &= -\,C^iT_iF + \hat{c}^\mu\partial_\mu F - 2\mathrm{i}\nabla_\mu\chi\sigma^\mu\bar\xi + 4\bar\lambda^i\bar\xi T_i\varphi\,. \end{aligned} \tag{9.2}$$

4. The extended BRST-transformations of the antifields are according to (2.23) obtained from an extended lagrangian which is now given by

$$L_{\mathrm{ext}} = L - (s_{\mathrm{ext}}\Phi^N)\Phi^*_N\,. \tag{9.3}$$

The passage from the formulation with auxiliary fields to the formulation without auxiliary fields is done by elimination of the auxiliary fields using the solution of their "extended equations of motion" derived from L_{ext} and setting the antifields D^*_i, F^*_s, $\bar{F}^*_s$ to zero. This amounts to the following identifications:

$$D^i \equiv -\mathrm{i}\delta^{ij}(\bar\varphi T_j\varphi + \lambda^*_j\xi + \bar\xi\bar\lambda^*_j)\,, \qquad D^*_i \equiv 0\,,$$

$$F \equiv -4\bar{\xi}\bar{\chi}^* , \qquad \bar{F} \equiv 4\chi^*\xi , \qquad F^* \equiv 0 \equiv \bar{F}^* . \tag{9.4}$$

These identifications reproduce the formulae given in section 2 und turn representatives of the cohomologies $H(s_{\mathrm{ext}}, \mathfrak{F})$ and $H(s_{\mathrm{ext}}|d)$ in the formulation with the auxiliary fields into representatives of the same cohomologies in the formulation without auxiliary fields (see, e.g., section 15 of [5]). From the cohomological perspective, this reflects that the auxiliary fields and their antifields give only rise to trivial pairs (see section 6.2). These trivial pairs are $(D_i^*, s_{\mathrm{ext}} D_i^*)$, $(F_s^*, s_{\mathrm{ext}} F_s^*)$, $(\bar{F}_s^*, s_{\mathrm{ext}} \bar{F}_s^*)$ and all their derivatives, for one has:

$$\begin{aligned}
s_{\mathrm{ext}} D_i^* &= \delta_{ij} D^j + \mathrm{i}\,(\bar{\varphi} T_i \varphi + \lambda_i^* \xi + \bar{\xi} \bar{\lambda}_i^*) + C^j f_{ji}{}^k D_k^* + \hat{c}^\mu \partial_\mu D_i^* \\
s_{\mathrm{ext}} F^* &= \frac{1}{4}\bar{F} - \chi^* \xi + C^i F^* T_i + \hat{c}^\mu \partial_\mu F^* \\
s_{\mathrm{ext}} \bar{F}^* &= \frac{1}{4} F + \bar{\xi}\bar{\chi}^* - C^i T_i \bar{F}^* + \hat{c}^\mu \partial_\mu \bar{F}^* .
\end{aligned} \tag{9.5}$$

9.2 Formulation with linearly realized supersymmetry

The standard formulation with linearly realized supersymmetry (often formulated in superspace) uses complete real vector multiplets in place of only the gauge potentials and the gauginos, and a higher gauge symmetry which gives rise to additional ghost fields. The linearized additional gauge transformations shift the additional fields of the vector multiplets by additional gauge parameters. As a consequence, the extended BRST transformations act on these fields as nonlinearly extended shift transformations involving the additional ghost fields. The extended BRST transformations of the antifields of the additional ghost fields are nonlinearly extended shift transformations involving the antifields of the additional fields of the vector multiplets. Hence, all additional fields, antifields and their derivatives give only rise to trivial pairs. Elimination of these trivial pairs reproduces the familiar Wess-Zumino gauged models described in section 9.1. If one also removes the auxiliary fields as outlined there, one ends up precisely with the formulation as in section 2. Let us show this explicitly for the abelian case with one vector multiplet and one chiral matter multiplet (linear multiplets need not be considered here because supersymmetry is already linearly realized on them). We denote the fields of the vector multiplet by V, ζ_α, Z, A'_μ, λ'_α and D' and those of the chiral multiplet by φ', χ'_α and F', where V, A'_μ and D' are real fields, while the other fields are complex. The ghost fields are denoted by Λ, Γ_α and Σ and are all complex (they form a chiral supersymmetry multiplet). The extended BRST transformations of these fields are

$$\begin{aligned}
s_{\mathrm{ext}} V &= \mathrm{i}(\Lambda - \bar{\Lambda}) + \xi\zeta + \bar{\zeta}\bar{\xi} + \hat{c}^\mu \partial_\mu V \\
s_{\mathrm{ext}} \zeta_\alpha &= \Gamma_\alpha + \hat{c}^\mu \partial_\mu \zeta_\alpha - \frac{1}{2} c^{\mu\nu} (\sigma_{\mu\nu}\zeta)_\alpha - (\sigma^\mu \bar{\xi})_\alpha (A'_\mu + \mathrm{i}\partial_\mu V) - \xi_\alpha Z \\
s_{\mathrm{ext}} Z &= \Sigma + \hat{c}^\mu \partial_\mu Z + 2\mathrm{i}\partial_\mu \zeta \sigma^\mu \bar{\xi} + 2\mathrm{i}\bar{\lambda}' \bar{\xi} \\
s_{\mathrm{ext}} A'_\mu &= -\partial_\mu(\Lambda + \bar{\Lambda}) + \hat{c}^\nu \partial_\nu A'_\mu - c_\mu{}^\nu A'_\nu - \mathrm{i}\,\xi \sigma_\mu \bar{\lambda}' + \mathrm{i}\,\lambda' \sigma_\mu \bar{\xi} + \mathrm{i}\partial_\mu(\xi\zeta - \bar{\zeta}\bar{\xi}) \\
s_{\mathrm{ext}} \lambda'_\alpha &= \hat{c}^\mu \partial_\mu \lambda'_\alpha - \frac{1}{2} c^{\mu\nu} (\sigma_{\mu\nu} \lambda')_\alpha - \mathrm{i}\xi_\alpha D' + 2(\sigma^{\mu\nu}\xi)_\alpha \partial_\mu A'_\nu \\
s_{\mathrm{ext}} D' &= \hat{c}^\mu \partial_\mu D' + \xi \sigma^\mu \partial_\mu \bar{\lambda}' + \partial_\mu \lambda' \sigma^\mu \bar{\xi}.
\end{aligned}$$

$$
\begin{aligned}
s_{\text{ext}}\varphi' &= -2\mathrm{i}\Lambda\varphi' + \hat{c}^{\mu}\partial_{\mu}\varphi' + \xi\chi' \\
s_{\text{ext}}\chi'_{\alpha} &= -2\mathrm{i}\Lambda\chi'_{\alpha} - 2\Gamma_{\alpha}\varphi' + \hat{c}^{\mu}\partial_{\mu}\chi'_{\alpha} - \frac{1}{2}c^{\mu\nu}(\sigma_{\mu\nu}\chi')_{\alpha} - 2\mathrm{i}(\sigma^{\mu}\bar{\xi})_{\alpha}\partial_{\mu}\varphi' + \xi_{\alpha}F' \\
s_{\text{ext}}F' &= -2\mathrm{i}\Lambda F' + 2\Gamma\chi' + 2\Sigma\varphi' + \hat{c}^{\mu}\partial_{\mu}F' - 2\mathrm{i}\partial_{\mu}\chi'\sigma^{\mu}\bar{\xi} \\
s_{\text{ext}}\Lambda &= \hat{c}^{\mu}\partial_{\mu}\Lambda + \mathrm{i}\xi\Gamma \\
s_{\text{ext}}\Gamma_{\alpha} &= \hat{c}^{\mu}\partial_{\mu}\Gamma_{\alpha} - \frac{1}{2}c^{\mu\nu}(\sigma_{\mu\nu}\Gamma)_{\alpha} + \xi_{\alpha}\Sigma - 2(\sigma^{\mu}\bar{\xi})_{\alpha}\partial_{\mu}\Lambda \\
s_{\text{ext}}\Sigma &= \hat{c}^{\mu}\partial_{\mu}\Sigma - 2\mathrm{i}\partial_{\mu}\Gamma\sigma^{\mu}\bar{\xi}\,.
\end{aligned}
\tag{9.6}
$$

The presence of the shift terms $\mathrm{i}(\Lambda - \bar{\Lambda})$, Γ_{α} and Σ in $s_{\text{ext}}V$, $s_{\text{ext}}\zeta_{\alpha}$ and $s_{\text{ext}}Z$ implies that $(V, s_{\text{ext}}V)$, $(\zeta_{\alpha}, s_{\text{ext}}\zeta_{\alpha})$, $(Z, s_{\text{ext}}Z)$ and their derivatives form indeed trivial pairs. The same holds of course for $(\bar{\zeta}_{\dot{\alpha}}, s_{\text{ext}}\bar{\zeta}_{\dot{\alpha}})$, $(\bar{Z}, s_{\text{ext}}\bar{Z})$ and their derivatives. It is straightforward to verify that the following identifications reproduce the s_{ext}-transformations of A_{μ}, φ, λ, D, χ, F and C given in (2.8), (2.13), (9.2) and (2.15) for the abelian case (with $C^{i}T_{i}\varphi \equiv -\mathrm{i}C\varphi$ etc):

$$
\begin{aligned}
&A_{\mu} = A'_{\mu}\,, \qquad \lambda_{\alpha} = \lambda'_{\alpha}\,, \qquad D = D'\,, \qquad C = -\Lambda - \bar{\Lambda} + \mathrm{i}\xi\zeta - \mathrm{i}\bar{\zeta}\bar{\xi}\,, \\
&\varphi = \mathrm{e}^{V}\varphi'\,, \qquad \chi_{\alpha} = \mathrm{e}^{V}(\chi'_{\alpha} + 2\zeta_{\alpha}\varphi')\,, \qquad F = \mathrm{e}^{V}(F' - 2Z\varphi' - 2\zeta\chi' - 2\zeta\zeta\varphi')\,.
\end{aligned}
\tag{9.7}
$$

The antifields of the additional fields give also only rise to trivial pairs, as one has

$$
\begin{aligned}
s_{\text{ext}}\Lambda^{*} &= \mathrm{i}V^{*} + \partial_{\mu}A^{*\mu\prime} + 2\partial_{\mu}\Gamma^{*}\sigma^{\mu}\bar{\xi} - 2\mathrm{i}(\varphi^{*\prime}\varphi' + \chi^{*\prime}\chi' + F^{*\prime}F') \\
s_{\text{ext}}\Gamma^{*\alpha} &= -\zeta^{*\alpha} - \mathrm{i}\xi^{\alpha}\Lambda^{*} - 2\mathrm{i}\partial_{\mu}\Sigma^{*}\sigma^{\mu}\bar{\xi} + 2\varphi'\chi^{*\alpha\prime} + 2\chi^{\alpha\prime}F^{*\prime} \\
s_{\text{ext}}\Sigma^{*} &= Z^{*} + \Gamma^{*}\xi + 2\varphi' F^{*\prime}\,.
\end{aligned}
\tag{9.8}
$$

The non-abelian case is analogous except that the extended BRST transformations of the fields receive further nonlinear contributions, and, as a consequence, the relation of primed and unprimed fields becomes more complicated (in particular A_{μ}, λ and D do not coincide anymore with their primed counterparts).

10. Other lagrangians

Even though our analysis was performed for lagrangians (2.2), it is not restricted to these. In particular it applies analogously to more general lagrangians which are of the form (7.6) in the formulation with auxiliary fields (with $\mathcal{D}_{\alpha}$ and $\bar{\mathcal{D}}_{\dot{\alpha}}$ realized off-shell on ordinary tensor fields, cf. comment c) in section 6.5), such as

$$
L = \mathcal{D}^{2}[G_{ij}(\varphi)\lambda^{i}\lambda^{j} + P(\varphi) + \bar{\mathcal{D}}^{2}K(\varphi, \bar{\varphi}, \phi)] + \text{c.c.} \tag{10.1}
$$

where $G_{ij}(\varphi)\lambda^{i}\lambda^{j}$, $P(\varphi)$ and $K(\varphi, \bar{\varphi}, \phi)$ are gauge invariant functions. (10.1) is a generalization of the lagrangian (2.2) which is still at most quadratic in derivatives but in general not power counting renormalizable as it contains terms such as $(G_{ij}(\varphi) + \text{c.c.})F^{i}_{\mu\nu}F^{j\mu\nu}$. The special choice $G_{ij}(\varphi) = \delta_{ij}/16$, $P(\varphi) = 0$ and $K(\varphi, \bar{\varphi}, \phi) = (\bar{\varphi}\varphi - \phi^{t}\phi)/32$ reproduces the lagrangians (2.2) after elimination of the auxiliary fields, up to a total divergence. Let us briefly indicate how one can apply our analysis and results to such models or even more general lagrangians (7.6) containing terms with more than two derivatives such as (7.22), assuming that supersymmetry is not spontaneously broken (see comment below) and that the field equations satisfy the standard regularity conditions (see [7]):

(i) The extended BRST transformations of the fields and constant ghosts are as in section 9.1 when one uses the formulation with auxiliary fields (recall that the deformations (7.6) do not change the gauge or supersymmetry transformations when one uses the formulation with the auxiliary fields, cf. comment (f) in section 7).

(ii) The extended BRST transformations of the antifields contain the Euler-Lagrange derivatives of the lagrangian with respect to the corresponding fields and are thus more involved than in the simple models (2.2). As a consequence the explicit form of the w-variables (see section 4) changes. A subset of these variables even may fail to be local in the strict sense of section 3 (because the algorithm [11] may not terminate anymore). This failure of locality does not happen for lagrangians (10.1)[19] but it will generically happen for more general lagrangians. In order to apply our analysis in the latter case one must relax the definition of local functions and forms accordingly. In particular this is necessary and natural when dealing with effective lagrangians containing terms such as (7.22) multiplied by free parameters (coupling constants). One may then use the definition that local functions and forms are formal power series in the free parameters, with each term of the series local in the strict sense, cf. the remarks on effective theories in [7].

(iii) The descent equations and lemma 1 hold also in the generalized models.

(iv) The gauge covariant algebra (5.9) does not change because it reflects the extended BRST transformations of the ghosts which do not change.

(v) The structure of the cohomological groups presented in section 6.1 does not change because $s_{\mathrm{ext}}R_a^*$ and $s_{\mathrm{ext}}C_{i_{\mathrm{f}}}^*$ do not change (this follows from the fact that the gauge transformations do not change). However, f_{ab}, f'_{ab}, $\tilde{H}^a$ and $\tilde{F}^{i_{\mathrm{f}}}$ can receive additional terms because of changes of the field equations.

(vi) Subsections 6.2, 6.3 and 6.4 evidently apply also to the generalized models.

(vii) $H(s_{\mathrm{susy}}, \mathfrak{T}_{\mathrm{inv}})$ is still given by lemma 11, except for the possible modifications of $\tilde{H}^a$ and $\tilde{F}^{i_{\mathrm{f}}}$ mentioned above. Indeed, the results for $g \leq 2$ derive as before from the results in section 6.1 and from eq. (B.29) which holds also in the generalized models (the proof of that equation applies also to the generalized models). The results for $g \geq 3$ are solely based on the algebra (5.9) and the structure of the supersymmetry multiplets which do not change.

[19] Lemma 3 still holds for these lagrangians but the proof of the lemma is more involved than for the simple lagrangians (2.2). Without going into detail I note that one may use the derivative order of the variables (rather than their dimension) to prove that the algorithm [11] terminates for lagrangians (10.1).

(viii) Lemmas 12 and 13 apply without modifications also to the generalized models.

(ix) The possible changes of lemma 14 are induced by the modifications of $\tilde{H}^a$ and $\tilde{F}^{i_{\mathrm{f}}}$ mentioned above and concern the representatives $f^{(2)}$, $f^{(3)}$, $f^{(4)}$ and $f^{(5)}$. $f^{(2)}$ and $f^{(5)}$ have counterparts also in the generalized models but their explicit form changes when $\tilde{H}^a$ receives additional terms (the additional terms cause changes of the X-functions occurring in $f^{(2)}$ and $f^{(5)}$; the existence of these representatives is still guaranteed by the vanishing of $H^g(s_{\mathrm{susy}}, \mathfrak{T}_{\mathrm{inv}})$ for $g > 4$). The existence and precise form of representatives $f^{(3)}$ and $f^{(4)}$ depends on the modifications of $\tilde{H}^a$ and $\tilde{F}^{i_{\mathrm{f}}}$ and varies from case to case. However, the modifications of $f^{(3)}$ do not concern consistent deformations, counterterms or anomalies because there are no representatives $f^{(3)}$ with ghost numbers 4 or 5. The representatives $f^{(4)}$ are mainly of interest for the deformation of free theories with lagrangians of the form (2.2).

Comment. In models with spontaneously broken supersymmetry the goldstino fields and their extended BRST transformations form trivial pairs (the extended BRST transformations of the goldstino fields substitute for the constant supersymmetry ghosts in the new jet coordinates $\{u^\ell, v^\ell, w^I\}$). As a consequence the structure of the extended BRST cohomology is essentially the same as its non-supersymmetric counterpart, provided one relaxes the definition of local functions and forms suitably, if necessary (see remarks above). The representatives of the cohomology are supersymmetrizations of their counterparts in the non-supersymmetric cohomology, similarly to the supersymmetrized actions constructed in [58, 59, 60].

11. Conclusion

The major advances of our analysis as compared to previous work are: (i) we have computed the cohomology in the space of all local forms rather than only in the restricted space of forms with bounded power-counting dimension; (ii) we have included linear multiplets in addition to super Yang-Mills multiplets and chiral multiplets. Furthermore we have computed the cohomology for all ghost numbers even though the results for $H^{g,4}(s_{\mathrm{ext}}|d)$, $g > 1$ are currently only of mathematical interest as no physical interpretation of these cohomological groups is known to date. Let us briefly summarize the results for the cohomological groups $H^{0,4}(s_{\mathrm{ext}}|d)$ and $H^{1,4}(s_{\mathrm{ext}}|d)$ which are most important for algebraic renormalization, candidate anomalies and supersymmetric consistent deformations of the models under study.

The results are particularly simple when the Yang-Mills gauge group is semisimple and no linear multiplets are present: then all representatives of $H^{0,4}(s_{\mathrm{ext}}|d)$ can be written in the form (7.6) (times the volume element, and up to antifield dependent terms in the formulation without auxiliary fields), and the representatives of $H^{1,4}(s_{\mathrm{ext}}|d)$ are exhausted by (7.8) (up to cohomologically trivial terms, respectively). Hence, for semisimple gauge group and in absence of linear multiplets, (i) all Poincaré invariant and $N = 1$ supersymmetric consistent deformations of the action which preserve the $N = 1$ supersymmetry algebra on-shell modulo gauge transformations can be constructed from standard superspace integrals and preserve the form of the gauge transformations and $N = 1$ supersymmetry transformations when one uses the auxiliary fields (accordingly in the formulation without auxiliary fields only the supersymmetry transformations of the fermion fields get deformed as one sees by elimination of the auxiliary fields); (ii) all counterterms that are gauge invariant, Poincaré invariant and $N = 1$ supersymmetric at least on-shell can be written even as off-shell invariants by means of the auxiliary fields and are constructible from standard superspace integrals; (iii) the consistent Poincaré invariant candidate gauge and supersymmetry anomalies are exhausted by supersymmetric generalizations of the well-known non-abelian chiral anomalies and these can be written in the universal form (7.8) whether or not one uses the auxiliary fields. These results are not restricted to lagrangians (2.2) but apply also to a general class of lagrangians and in particular to effective super-Yang-Mills theories with semisimple gauge group, see section 10.

When the Yang-Mills gauge group contains abelian factors or when linear multiplets are present, there are a number of additional cohomology classes of $H^{0,4}(s_{\rm ext}|d)$ and $H^{1,4}(s_{\rm ext}|d)$ whose representatives cannot be written as in (7.6) or (7.8). We have computed them explicitly for the simple lagrangians (2.2) because they are particularly relevant to the general classification of supersymmetric consistent interactions which naturally starts off from free models with lagrangians (2.2). The antifield independent parts of the additional representatives are given in equations (7.1)–(7.5) and (7.9)–(7.15). From among these only the candidate anomalies (7.15) and a few special solutions are off-shell invariants or can be written as off-shell invariants by means of the auxiliary fields. The special off-shell solutions are particular representatives (7.5) given by the Fayet-Iliopoulos terms (7.18) and the Chern-Simons type representatives (7.19), analogous representatives (7.13) with ghost number 1 given in (7.23) and (7.24), and particular representatives (7.14) given in (7.27) and (7.29). All other functions (7.1)–(7.5) and forms (7.9)–(7.14) are accompanied by antifield dependent terms and may look somewhat different when one uses a more general lagrangian (some of them might even disappear), cf. section 10. The interaction terms contained in (7.1)–(7.5) include, among other things, Yang-Mills, Freedman-Townsend and Chapline-Manton vertices, and Noether couplings of gauge fields to gauge invariant conserved currents that are supersymmetric up to trivial currents.

Finally we remark that the results derived in this paper are characteristic of theories with a particular supersymmetry multiplet structure on tensor fields which we have termed "QDS structure" (see [61] and proof of lemma 11). Indeed, when one reviews the derivation of the results one observes that, essentially, they can be put down to three central ingredients: the characteristic cohomology of the field equations in form-degrees smaller than 3 (see section 6.1), standard Lie algebra cohomology (see section 6.4), and what we have termed supersymmetry algebra cohomology (see section 6.5). While the characteristic cohomology and Lie algebra cohomology are not affected by the supersymmetry multiplet structure, this structure is decisive for the supersymmetry algebra cohomology. The QDS structure underlies the results for ghost numbers larger than 2 in lemma 11 and these results hold analogously for all models with QDS structure as can be inferred from the proof of the lemma.

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A. Conventions and useful formulae

Minkowski metric, ε-tensors:

$$\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1), \qquad \varepsilon^{\mu\nu\rho\sigma} = \varepsilon^{[\mu\nu\rho\sigma]}, \qquad \varepsilon^{0123} = 1,$$
$$\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}, \qquad \varepsilon^{\dot\alpha\dot\beta} = -\varepsilon^{\dot\beta\dot\alpha}, \qquad \varepsilon^{12} = \varepsilon^{\dot 1\dot 2} = 1,$$
$$\varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta_\alpha^\beta = \mathrm{diag}(1,1), \qquad \varepsilon_{\dot\alpha\dot\gamma}\varepsilon^{\dot\gamma\dot\beta} = \delta_{\dot\alpha}^{\dot\beta} = \mathrm{diag}(1,1)$$

σ-matrices:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\bar\sigma^{\mu\,\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}\,, \qquad \sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)\,, \qquad \bar\sigma^{\mu\nu} = \frac{1}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)$$

Raising, lowering, contraction of spinor indices:

$$\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta\,, \qquad \psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta\,, \qquad \bar\psi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}\,,$$
$$\bar\psi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}, \qquad \psi\chi = \psi^\alpha\chi_\alpha\,, \qquad \bar\psi\bar\chi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha}\,.$$

Lorentz vector indices in spinor notation:

$$V_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}V_\mu$$

Grassmann parity:

$$|X_{\alpha_1\ldots\alpha_n}^{\dot\alpha_1\ldots\dot\alpha_m}| = m + n + \mathrm{gh}(X) + \text{form-degree}(X) \pmod 2\,.$$

Complex conjugation:

$$\overline{XY} = (-)^{|X|\,|Y|}\bar X\ \bar Y$$

Hodge dual:

$$\star(dx^{\mu_1}\cdots dx^{\mu_p}) = \frac{1}{(n-p)!}dx^{\nu_1}\cdots dx^{\nu_{n-p}}\varepsilon_{\nu_1\ldots\nu_{n-p}}{}^{\mu_1\ldots\mu_p}\,.$$

Symmetrization and antisymmetrization of indices:

$$T_{(a_1\ldots a_n)} = \frac{1}{n!}\sum_{\pi\in S_n} T_{a_{\pi(1)}\ldots a_{\pi(n)}}, \qquad T_{[a_1\ldots a_n]} = \frac{1}{n!}\sum_{\pi\in S_n}(-)^{sign(\pi)}T_{a_{\pi(1)}\ldots a_{\pi(n)}}\,.$$

Frequently used functions and operators:

$$\Xi_{\mu\nu} = -\frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}\hat c^\rho\hat c^\sigma\,, \qquad \Xi_\mu = -\frac{1}{6}\varepsilon_{\mu\nu\rho\sigma}\hat c^\nu\hat c^\rho\hat c^\sigma\,, \qquad \Xi = -\frac{1}{24}\varepsilon_{\mu\nu\rho\sigma}\hat c^\mu\hat c^\nu\hat c^\rho\hat c^\sigma$$
$$\Theta = \hat c^\mu\xi\sigma_\mu\bar\xi\,, \qquad \vartheta^\alpha = \hat c^\mu(\bar\xi\bar\sigma_\mu)^\alpha\,, \qquad \bar\vartheta^{\dot\alpha} = \hat c^\mu(\bar\sigma_\mu\xi)^{\dot\alpha}$$
$$\hat C^i = C^i + A^i_\mu\hat c^\mu$$
$$\hat R = R + \mathrm{i}Q_\mu\hat c^\mu + \frac{\mathrm{i}}{2}B_{\mu\nu}\hat c^\nu\hat c^\mu$$
$$\tilde H^a = \mathrm{i}(\hat{\bar\psi}{}^a\bar\xi - \xi\hat\psi^a) + \hat c^\mu\hat H^a_\mu$$

$$
\begin{aligned}
&= \mathrm{i}(\bar\psi^a\bar\xi - \xi\psi^a) + \hat{c}^\mu H^a_\mu + \delta^{ab}\left(\vartheta\psi^*_b - \bar\psi^*_b\bar\vartheta + 2\,\Xi_{\mu\nu}B^{*\mu\nu}_b - \Xi_\mu Q^{*\mu}_b - \mathrm{i}\,\Xi\, R^*_b\right)\\
\tilde{F}^{i_\mathrm{f}} &= \hat{c}^\mu\left(\xi\sigma_\mu\bar{\hat\lambda}{}^{i_\mathrm{f}} + \hat\lambda^{i_\mathrm{f}}\sigma_\mu\bar\xi\right) + \frac{1}{4}\hat{c}^\mu\hat{c}^\nu\varepsilon_{\mu\nu\rho\sigma}\hat{F}^{i_\mathrm{f}\rho\sigma}\\
&= \vartheta\lambda^{i_\mathrm{f}} - \bar\lambda^{i_\mathrm{f}}\bar\vartheta - \Xi_{\mu\nu}F^{i_\mathrm{f}\mu\nu} + \delta^{i_\mathrm{f}j_\mathrm{f}}(-2\,\Xi_{\mu\nu}\,\xi\sigma^{\mu\nu}\lambda^*_{j_\mathrm{f}} - 2\,\Xi_{\mu\nu}\,\bar\lambda^*_{j_\mathrm{f}}\bar\sigma^{\mu\nu}\bar\xi + \Xi_\mu A^{*\mu}_{j_\mathrm{f}} + \Xi\, C^*_{j_\mathrm{f}})\\
\mathcal{F}^i &= \mathrm{i}\hat{c}^\mu\left(\xi\sigma_\mu\bar{\hat\lambda}{}^i - \hat\lambda^i\sigma_\mu\bar\xi\right) + \frac{1}{2}\hat{c}^\mu\hat{c}^\nu\hat{F}^i_{\mu\nu}\\
&= -\mathrm{i}\vartheta\lambda^i - \mathrm{i}\bar\vartheta\bar\lambda^i + \frac{1}{2}\hat{c}^\mu\hat{c}^\nu F^i_{\mu\nu}\\
\mathcal{H}^a &= -2\hat{c}^\mu\xi\sigma_\mu\bar\xi\phi^a - \mathrm{i}\hat{c}^\mu\hat{c}^\nu\left(\xi\sigma_{\mu\nu}\hat\psi^a - \bar{\hat\psi}{}^a\bar\sigma_{\mu\nu}\bar\xi\right) - \frac{\mathrm{i}}{6}\hat{c}^\mu\hat{c}^\nu\hat{c}^\rho\varepsilon_{\mu\nu\rho\sigma}\hat{H}^{a\sigma}\\
&= -2\Theta\phi^a + 2\,\Xi_{\mu\nu}\left(\xi\sigma^{\mu\nu}\psi^a + \bar\psi^a\bar\sigma^{\mu\nu}\bar\xi\right) - \mathrm{i}\,\Xi_\mu H^{a\mu}\\
\mathcal{O} &= 4\Theta - 4\,\Xi_{\mu\nu}(\xi\sigma^{\mu\nu}\mathcal{D} + \bar\xi\bar\sigma^{\mu\nu}\bar{\mathcal{D}}) - \frac{\mathrm{i}}{2}\Xi_\mu\sigma^\mu_{\alpha\dot\alpha}[\mathcal{D}^\alpha,\bar{\mathcal{D}}^{\dot\alpha}]\\
\mathcal{P} &= -16\Xi_{\mu\nu}\bar\xi\bar\sigma^{\mu\nu}\bar\xi - 4\mathrm{i}\Xi_\mu\xi\bar\sigma^\mu\mathcal{D} + \Xi\mathcal{D}^2\\
&= 4\mathrm{i}\vartheta\vartheta - 4\mathrm{i}\Xi_\mu\xi\bar\sigma^\mu\mathcal{D} + \Xi\mathcal{D}^2
\end{aligned}
$$

$\mathcal{D}_\alpha$-transformations:

$$
\begin{aligned}
&\mathcal{D}_\alpha\hat\lambda^i_\beta = -\hat{F}^{i(+)}_{\alpha\beta} + \varepsilon_{\alpha\beta}\,\delta^{ij}\,\bar\varphi T_j\varphi\,, \qquad \mathcal{D}_\alpha\bar{\hat\lambda}{}^i_{\dot\beta} = 0\,,\\
&\mathcal{D}_\alpha\hat{F}^{i(+)}_{\beta\gamma} = -2\delta^{ij}\,\varepsilon_{\alpha(\beta}\,\bar\varphi T_j\hat\chi_{\gamma)}\,, \qquad \mathcal{D}_\alpha\hat{F}^{i(-)}_{\dot\beta\dot\gamma} = -2\mathrm{i}\hat\nabla_{\alpha(\dot\beta}\bar{\hat\lambda}{}^i_{\dot\gamma)}\\
&\mathcal{D}_\alpha\phi = \hat\psi_\alpha\,, \quad \mathcal{D}_\alpha\hat\psi_\beta = 0\,, \qquad \mathcal{D}_\alpha\bar{\hat\psi}_{\dot\beta} = -\hat{H}_{\alpha\dot\beta} - \mathrm{i}\hat\nabla_{\alpha\dot\beta}\phi\,, \qquad \mathcal{D}_\alpha\hat{H}_\beta{}^{\dot\beta} = -\mathrm{i}\hat\nabla_{(\alpha}{}^{\dot\beta}\psi_{\beta)}\\
&\mathcal{D}_\alpha\varphi = \hat\chi_\alpha\,, \quad \mathcal{D}_\alpha\bar\varphi = 0\,, \qquad \mathcal{D}_\alpha\hat\chi_\beta = 0\,, \qquad \mathcal{D}_\alpha\bar{\hat\chi}_{\dot\beta} = -2\mathrm{i}\hat\nabla_{\alpha\dot\beta}\bar\varphi
\end{aligned}
$$

B. Proofs of the lemmas

Proof of lemma 1. First we show that (3.1) implies descent equations of the standard form:

$$
s_{\mathrm{ext}}\omega^{g,4} + d\omega^{g+1,3} = 0\,, \qquad s_{\mathrm{ext}}\omega^{g+1,3} + d\omega^{g+2,2} = 0\,, \qquad \ldots \qquad s_{\mathrm{ext}}\omega^{g+4-m,m} = 0\,, \tag{B.1}
$$

for some local forms $\omega^{g+4-p,p}$ and some form-degree m which is not a priori known but turns out to be necessarily 0 whenever $\omega^{g,4}$ is nontrivial (see below). The descent equations follow from the so-called algebraic Poincaré lemma (cf. section 4.5 of [7]) which describes the cohomology $H(d)$ (locally) in the jet space associated with the fields, antifields and the constant ghosts. It states that $H^p(d)$ vanishes locally[20] in form-degrees $p = 1, 2, 3$ and that $H^0(d)$ is represented by constants which means in the present case that $H^0(d)$ is given by $\mathfrak{C}$, the vector space of polynomials in the constant ghosts. One derives (B.1) in the usual manner: acting with s_{ext} on an equation $s_{\mathrm{ext}}\omega^{g+4-p,p} + d\omega^{g+5-p,p-1} = 0$ gives $d(s_{\mathrm{ext}}\omega^{g+5-p,p-1}) = 0$. If $p - 1 > 0$, the algebraic Poincaré lemma guarantees that there is a local form $\omega^{g+6-p,p-2}$ such that $s_{\mathrm{ext}}\omega^{g+5-p,p-1} + d\omega^{g+6-p,p-2} = 0$. If $p - 1 = 0$, then the algebraic Poincaré lemma alone only allows one to conclude $s_{\mathrm{ext}}\omega^{g+4,0} = M$ for some $M \in \mathfrak{C}$. Nevertheless, one can assume $M = 0$ without loss of generality, for the following

[20]Global aspects are left out of consideration in this work. They may be studied along the lines of [62, 9].

reason: an $s_{\rm ext}$-exact function which depends only on the constant ghosts is necessarily the $s_{\rm ext}$-transformation of a function which also depends only on the constant ghosts because the $s_{\rm ext}$-transformations of the fields and antifields do not contain parts depending only on the constant ghosts; hence, $s_{\rm ext}\omega^{g+4,0} = M$ implies $M = s_{\rm ext}N$ for some $N \in \mathfrak{C}$. The descent equations (B.1) then hold with $\omega^{g+4,0} - N$ in place of $\omega^{g+4,0}$ as one has $s_{\rm ext}(\omega^{g+4,0} - N) = 0$ and $s_{\rm ext}\omega^{g+3,1} + d(\omega^{g+4,0} - N) = 0$.

$H(s_{\rm ext} + d)$ enters in the usual manner by writing (B.1) into the compact form

$$(s_{\rm ext} + d)\tilde{\omega} = 0\,, \qquad \tilde{\omega} = \sum_{p=m}^{4} \omega^{g+4-p,p}\,. \tag{B.2}$$

Using standard arguments (see [63, 10, 7]) one deduces that $H(s_{\rm ext} + d)$ is isomorphic to $H^{*,4}(s_{\rm ext}|d) \oplus H(s_{\rm ext}, \mathfrak{C})$.

The isomorphism between $H(s_{\rm ext} + d)$ and $H(s_{\rm ext}, \mathfrak{F})$ rests on the fact that $(s_{\rm ext} + d)$ arises from $s_{\rm ext}$ on all variables except for x^μ simply through the substitution $c^\mu \to c^\mu + dx^\mu$.[21] Apart from this substitution, the only difference between $(s_{\rm ext} + d)$ and $s_{\rm ext}$ is that one has $(s_{\rm ext} + d)x^\mu = dx^\mu$ but $s_{\rm ext}x^\mu = 0$. However, this difference does not matter because x^μ drops essentially from $H(s_{\rm ext}+d)$. This is seen after changing variables from c^μ to $\tilde{c}^\mu = c^\mu + dx^\mu - x^\nu c_\nu{}^\mu$ (with all other variables left unchanged). Using the new variables, (x^μ, dx^μ) become trivial pairs which drop from $H(s_{\rm ext}+d)$ by the standard arguments (see section 6.2) because the x^μ and dx^μ form "$(s_{\rm ext} + d)$-doublets" due to $(s_{\rm ext} + d)x^\mu = dx^\mu$, and the $(s_{\rm ext} + d)$-transformations of the other variables do not involve x^μ or dx^μ when expressed in terms of the new variables (one has $s_{\rm ext} + d = \tilde{c}^\mu\partial_\mu + \cdots$, except on x^μ).

Hence, one has $H(s_{\rm ext} + d) \simeq H(s_{\rm ext} + d, \tilde{\mathfrak{F}})$ where $\tilde{\mathfrak{F}}$ is the analog of $\mathfrak{F}$, with $\tilde{c}^\mu$ in place of $\hat{c}^\mu$. The isomorphism $H(s_{\rm ext} + d, \tilde{\mathfrak{F}}) \simeq H(s_{\rm ext}, \mathfrak{F})$ is now evident, since $(s_{\rm ext} + d)$ acts in $\tilde{\mathcal{F}}$ exactly in the same manner as $s_{\rm ext}$ acts in $\mathfrak{F}$ (modulo the substitution $\tilde{c}^\mu \to \hat{c}^\mu$). $\square$

Proof of lemma 2. $[\partial_\mu, \partial_\nu] = 0$ implies the following identities:

$$\begin{aligned}
[\partial_+^+, \partial_+^-]Z_n^m &= [\partial_+^+, \partial_-^+]Z_n^m = [\partial_-^-, \partial_+^-]Z_n^m = [\partial_-^-, \partial_-^+]Z_n^m = 0\,,\\
[\partial_+^+, \partial_-^-]Z_n^m &= (m + n + 2)\Box Z_n^m\,,\\
[\partial_-^+, \partial_+^-]Z_n^m &= (m - n)\Box Z_n^m\,,\\
\partial_+^-\partial_-^+ Z_n^m &= \frac{1}{2}n(m+2)\,\Box Z_n^m + \partial_+^+\partial_-^- Z_n^m\,,
\end{aligned} \tag{B.3}$$

These identities can be used to construct a basis for the derivatives of a field or antifield: every kth order derivative can be expressed as a linear combination of the components of polynomials of degree k in the operations ∂_+^+, ∂_-^+, ∂_+^-, ∂_-^- applied to the field or antifield; using (B.3) one can construct an appropriate basis of these polynomials; the u's, linearized v's, and $w_{(0)}$'s just provide such a basis for the derivatives of all fields and antifields. For instance, consider a scalar field ϕ and its antifield ϕ^*. Using (B.3) one verifies straightforwardly that bases for all derivatives of ϕ and ϕ^* are $\cup_{p,q}\Box^p(\partial_+^+)^q\phi$ and $\cup_{p,q}\Box^p(\partial_+^+)^q\phi^*$,

[21]This is a direct consequence of the presence of spacetime translations in $s_{\rm ext}$: $s_{\rm ext}$ contains the translational piece $c^\mu\partial_\mu$ (except on x^μ) and thus $(s_{\rm ext} + d)$ contains $(c^\mu + dx^\mu)\partial_\mu$.

respectively. $\cup_{p,q}\Box^p(\partial_+^+)^q\phi^*$ is contained in $\{u^\ell\}$. One has $s_{\rm ext}\phi^* = -\Box\phi + \cdots$ and thus $s_{\rm ext}\Box^p(\partial_+^+)^q\phi^* = -\Box^{p+1}(\partial_+^+)^q\phi + \cdots$. Hence the set of the linearized variables $v^\ell = s_{\rm ext}u^\ell$ contains a basis for all derivatives of ϕ except for the $(\partial_+^+)^q\phi$. The latter are contained in $\{w^I_{(0)}\}$. Analogous arguments apply to the other fields and antifields, see [12] for further details. $\Box$

Proof of lemma 3. We only need to prove that the algorithm in section 2 of [11] produces w's that are local functions (eq. (4.1) holds by the algorithm). This can be done along the lines of section 3 of [11] using the following dimension assignments:

X	$R, \hat{c}^\mu$	ξ	$C, Q_\mu, c^{\mu\nu}$	$A_\mu, B_{\mu\nu}, \phi, \varphi, \partial_\mu$	λ, ψ, χ	Φ^*_A
$\dim(X)$	-1	$-1/2$	0	1	$3/2$	$4-\dim(\Phi^A)\,.$

(B.4)

With these assignments we have: (i) $s_{\rm ext}$ has dimension 0, (ii) all u's, v's, $w_{(0)}$'s with non-positive dimension have positive ghost numbers and dimensions ≥ -1, (iii) all u's, v's, $w_{(0)}$'s with negative ghost numbers have dimensions $\geq 5/2$. Because of (i) and since $s_{\rm ext}$ has ghost number 1, each w-variable constructed by means of the algorithm in [11] has a definite dimension and a definite ghost number. Properties (ii) and (iii) imply that there are only finitely many monomials in the u's, v's and $w_{(0)}$'s with a given dimension and a given ghost number. Hence, each w-variable is a finite sum of such monomials and thus a local function. $\Box$

Proof of lemma 5. $s_{\rm ext}^2 = 0$ and (5.6) imply

$$\begin{aligned} 0 = s_{\rm ext}^2 f(\hat{T}) &= (s_{\rm ext}\mathcal{C}^M)\Delta_M f(\hat{T}) + (-)^{|M|+1}\mathcal{C}^M\mathcal{C}^N\Delta_N\Delta_M f(\hat{T}) \\ &= (s_{\rm ext}\mathcal{C}^M)\Delta_M f(\hat{T}) + \frac{1}{2}(-)^{|M|+1}\mathcal{C}^M\mathcal{C}^N[\Delta_N,\Delta_M\}f(\hat{T}) \end{aligned} \tag{B.5}$$

where $|M|$ is the Grassmann parity of Δ_M and $[\ ,\ \}$ is the graded commutator:

$$[\Delta_N,\Delta_M\} := \Delta_N\Delta_M - (-)^{|N|\,|M|}\Delta_M\Delta_N\,.$$

The transformations of the $\mathcal{C}$'s have the form

$$s_{\rm ext}\mathcal{C}^P = \frac{1}{2}(-)^{|M|+1}\mathcal{C}^M\mathcal{C}^N\mathcal{F}_{NM}{}^P(\hat{T}) \tag{B.6}$$

for functions $\mathcal{F}_{NM}{}^P(\hat{T})$ which can be read off from (5.1), (3.4), (2.19), (2.20) and (2.21). Since (B.5) holds for all functions $f(\hat{T})$ and since the $\mathcal{C}^M$ are independent variables, we conclude

$$[\Delta_N,\Delta_M\} = -\mathcal{F}_{NM}{}^P(\hat{T})\Delta_P\,. \tag{B.7}$$

This yields (5.9) when spelled out explicitly. $\Box$

Proof of lemma 6. The lemma is proved exactly as the corresponding results in [5] (see also [15]). We shall therefore only sketch the basic ideas and refer to [5] for details.

The results for $k > 3$ follow from the general theorems 8.3, or 10.1 and 10.2 in [5] because the models under study have Cauchy order 3 and reducibility order 1 in the terminology used there (the proofs of these theorems given in [5] apply also in presence of

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the constant ghosts because the latter are inert to both δ and d). To prove the results for $k=3$ and $k=2$ we first consider the linearized theory and derive $H^4_k(\delta^{(0)}|d)$ for $k=3$ and $k=2$ where $\delta^{(0)}$ is the Koszul-Tate differential of the linearized models (the $\delta^{(0)}$-transformations arise from (6.2) by linearization). The cocycle condition in $H^4_k(\delta^{(0)}|d)$ is $\delta^{(0)}\omega^4_k + d\omega^3_{k-1} = 0$ and reads in dual notation

$$\delta^{(0)} f_k + \partial_\mu f^\mu_{k-1} = 0, \tag{B.8}$$

where we used $\omega^4_k = d^4x f_k$ and $\omega^3_{k-1} = (1/6)dx^\mu dx^\nu dx^\rho \varepsilon_{\mu\nu\rho\sigma} f^\sigma_{k-1}$, i.e., f_k and f^σ_{k-1} are local functions (rather than forms) with antifield number k and $k-1$, respectively. One now considers the Euler-Lagrange derivatives of (B.8) with respect to the various fields and antifields. In particular, the Euler-Lagrange derivatives with respect to R^*_a, $Q^{*\mu}_a$, $B^{*\mu\nu}_a$, $B^a_{\mu\nu}$, C^*_i and $A^{*\mu}_i$ yield the following equations, respectively:

$$\delta^{(0)}\,\frac{\hat\partial f_k}{\hat\partial R^*_a} = 0 \tag{B.9}$$

$$\delta^{(0)}\,\frac{\hat\partial f_k}{\hat\partial Q^{*\mu}_a} = \mathrm{i}\,\partial_\mu\,\frac{\hat\partial f_k}{\hat\partial R^*_a} \tag{B.10}$$

$$\delta^{(0)}\,\frac{\hat\partial f_k}{\hat\partial B^{*\mu\nu}_a} = -\,\partial_\mu\,\frac{\hat\partial f_k}{\hat\partial Q^{*\nu}_a} + \partial_\nu\,\frac{\hat\partial f_k}{\hat\partial Q^{*\mu}_a} \tag{B.11}$$

$$\delta^{(0)}\,\frac{\hat\partial f_k}{\hat\partial B^a_{\mu\nu}} = \frac{1}{4}\delta_{ab}\,\varepsilon^{\mu\nu\rho\sigma}\varepsilon_\rho{}^{\lambda\kappa\tau}\partial_\sigma\partial_\lambda\,\frac{\hat\partial f_k}{\hat\partial B^{*\kappa\tau}_b} \tag{B.12}$$

$$\delta^{(0)}\,\frac{\hat\partial f_k}{\hat\partial C^*_i} = 0 \tag{B.13}$$

$$\delta^{(0)}\,\frac{\hat\partial f_k}{\hat\partial A^{*\mu}_i} = \partial_\mu\,\frac{\hat\partial f_k}{\hat\partial C^*_i}\,. \tag{B.14}$$

We discuss first the case $k=3$. $\hat\partial f_3/\hat\partial R^*_a$ has antifield number 0, i.e., it does not depend on antifields. (B.9) is thus trivially satisfied and imposes no condition in the case $k=3$. $\hat\partial f_3/\hat\partial Q^{*\mu}_a$ has antifield number 1 and thus $\delta^{(0)}\hat\partial f_3/\hat\partial Q^{*\mu}_a$ vanishes on-shell in the linearized theory. Hence (B.10) imposes $\partial_\mu(\hat\partial f_3/\hat\partial R^*_a) \approx^{(0)} 0$ where $\approx^{(0)}$ is equality on-shell in the linearized theory, i.e., $\hat\partial f_3/\hat\partial R^*_a$ are cocycles of the characteristic cohomology (= cohomology of d on-shell) of the linearized theory in form-degree 0. According to theorems 8.1 and 8.2 of [5] (or theorem 6.2 of [7]), the vanishing of $H^4_4(\delta^{(0)}|d)$ implies that the characteristic cohomology of the linearized theory is in form-degree 0 represented just by constants, which in our case are functions of the constant ghosts. This gives

$$\frac{\hat\partial f_3}{\hat\partial R^*_a} = k^a(c,\xi,\bar\xi) + \delta^{(0)} g^a_1\,, \tag{B.15}$$

for some local functions g^a_1 with antifield number 1. Using (B.15) in (B.10) and proceeding then as in section 9 of [5], one eventually obtains

$$f_3 = k^a(c,\xi,\bar\xi) R^*_a + \delta^{(0)} g_4 + \partial_\mu g^\mu_3\,, \tag{B.16}$$

for some local functions g_4 and g_3^μ with antifield numbers 4 and 3, respectively. Hence, $H_3^4(\delta^{(0)}|d)$ is represented just by the 4-forms $k^a(c,\xi,\bar\xi)R_a^* d^4x$. Owing to $\delta R_a^* = \delta^{(0)} R_a^*$, this result for $H_3^4(\delta^{(0)}|d)$ extends straightforwardly to $H_3^4(\delta|d)$ which proves the assertion in (6.3) for $k=3$.

The case $k=2$ can be treated analogously. $\hat\partial f_2/\hat\partial R_a^*$ vanishes (f_2 has antifield number 2 and thus cannot depend on R_a^*) while $\hat\partial f_2/\hat\partial Q_a^{*\mu}$ and $\hat\partial f_2/\hat\partial C_i^*$ do not depend on antifields. Hence (B.9), (B.10) and (B.13) are trivially satisfied and impose no condition for $k=2$. (B.14) imposes that $\hat\partial f_2/\hat\partial C_i^*$ are cocycles of the characteristic cohomology of the linearized theory in form-degree 0. Analogously to (B.15) we conclude

$$\frac{\hat\partial f_2}{\hat\partial C_i^*} = k^i(c,\xi,\bar\xi) + \delta^{(0)} g_1^i\,, \tag{B.17}$$

for some local functions g_1^i with antifield number 1. (B.11) imposes that $dx^\mu \hat\partial f_2/\hat\partial Q_a^{*\mu}$ are cocycles of the characteristic cohomology of the linearized theory in form-degree 1. The latter is isomorphic to $H_3^4(\delta^{(0)}|d)$ and this isomorphism is established through descent equations for $\delta^{(0)}$ and d (again, see theorems 8.1 and 8.2 of [5] or theorem 6.2 of [7], and the proofs of these theorems). Using the result for $H_3^4(\delta^{(0)}|d)$ which we just derived and the fact that the descent equations for $\delta^{(0)}$ and d relate $R_a^* d^4x$ to the 1-forms $dx^\mu H_{a\mu}$, we conclude

$$\frac{\hat\partial f_2}{\hat\partial Q_a^{*\mu}} = k^{ab}(c,\xi,\bar\xi) H_{b\mu} + \delta^{(0)} g^a_{\mu,1} + \partial_\mu g_0^a\,, \tag{B.18}$$

for some local functions $g^a_{\mu,1}$ and g_0^a with antifield numbers 1 and 0, respectively. Inserting (B.18) in (B.11) and using that $\partial_\mu H_{a\nu} - \partial_\nu H_{a\mu} = \varepsilon_{\mu\nu\rho\sigma}\delta^{(0)} B_a^{*\rho\sigma}$, one obtains from (B.11) (owing to the acyclicity of $\delta^{(0)}$ in positive antifield numbers, see, e.g., section 5 of [7]):

$$\frac{\hat\partial f_2}{\hat\partial B_a^{*\mu\nu}} = -k^{ab}(c,\xi,\bar\xi)\varepsilon_{\mu\nu\rho\sigma} B_b^{*\rho\sigma} - \partial_\mu g^a_{\nu,1} + \partial_\nu g^a_{\mu,1} + \delta^{(0)} g^a_{\mu\nu,2}\,, \tag{B.19}$$

for some local functions $g^a_{\mu\nu,2}$ with antifield number 2. Using (B.19) in (B.12), the latter gives (again owing to the acyclicity of $\delta^{(0)}$ in positive antifield numbers):

$$\frac{\hat\partial f_2}{\hat\partial B^a_{\mu\nu}} = \frac{1}{2}\delta_{ab}\, k^{bc}\varepsilon^{\mu\nu\rho\sigma}\partial_\sigma Q^*_{c\rho} - \frac{3}{2}\delta_{ab}\,\partial_\rho \partial^{[\rho} g^{\mu\nu]b}{}_{,2} + \delta^{(0)} g^{\mu\nu}_{a,3}\,, \tag{B.20}$$

for some local functions $g^{\mu\nu}_{a,3}$ with antifield number 3. Using (B.17) through (B.20) and proceeding then as in section 9 of [5], one obtains

$$\begin{aligned} f_2 = {} & k^i(c,\xi,\bar\xi) C_i^* + \frac{1}{2}\Big[k^{ab}(c,\xi,\bar\xi) - k^{ba}(c,\xi,\bar\xi)\Big] Q_a^{*\mu} H_{b\mu} - \\ & - \frac{1}{2} k^{ab}(c,\xi,\bar\xi)\varepsilon_{\mu\nu\rho\sigma} B_a^{*\mu\nu} B_b^{*\rho\sigma} + \delta^{(0)} g_3 + \partial_\mu g_2^\mu\,, \end{aligned} \tag{B.21}$$

for some local functions g_3 and g_2^μ with antifield numbers 3 and 2, respectively. Hence, $H_2^4(\delta^{(0)}|d)$ is represented by the 4-forms $k^i(c,\xi,\bar\xi)C_i^* d^4x$ and $k^{[ab]}(c,\xi,\bar\xi) f_{ab} d^4x$ with f_{ab} as given in the lemma. $k^{[ab]}(c,\xi,\bar\xi) f_{ab} d^4x$ is a cocycle of $H_2^4(\delta|d)$ owing to $\delta Q_a^{*\mu} = \delta^{(0)} Q_a^{*\mu}$

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and $\delta B_a^{*\mu\nu} = \delta^{(0)} B_a^{*\mu\nu}$. In contrast, $\delta^{(0)} C_i^*$ and δC_i^* do not coincide except for $i = i_{\rm f}$. The nonlinear terms in δC_i^* for other i obstruct the completability of these terms to cocycles of $H_2^4(\delta|d)$, see section 13 of [5]. This yields the result for $k = 2$ asserted in (6.3).

To prove (6.4) we take the Euler-Lagrange derivative of $R_a^* k^a(c,\xi,\bar{\xi}) = \delta g_4 + \partial_\mu g_3^\mu$ (which is equivalent to $R_a^* k^a(c,\xi,\bar{\xi}) d^4x \sim 0$) with respect to R_a^*. The result is

$$k^a(c,\xi,\bar{\xi}) = -\delta \, \frac{\hat{\partial} g_4}{\hat{\partial} R_a^*} \,. \tag{B.22}$$

The presence of δ implies that the right hand side contains only terms that are at least linear in fields or antifields, unless it vanishes. Hence both sides of (B.22) must vanish which gives $k^a(c,\xi,\bar{\xi}) = 0$. (6.5) can be proved in an analogous manner. The Euler-Lagrange derivatives of $C_{i_{\rm f}}^* k^{i_{\rm f}}(c,\xi,\bar{\xi}) + f_{ab} k^{[ab]}(c,\xi,\bar{\xi}) = \delta g_3 + \partial_\mu g_2^\mu$ with respect to $C_{i_{\rm f}}^*$ and $Q_a^{*\mu}$ read, respectively:

$$k^{i_{\rm f}}(c,\xi,\bar{\xi}) = \delta \, \frac{\hat{\partial} g_3}{\hat{\partial} C_{i_{\rm f}}^*} \,, \tag{B.23}$$

$$H_{b\mu} k^{[ab]}(c,\xi,\bar{\xi}) = \delta \, \frac{\hat{\partial} g_3}{\hat{\partial} Q_a^{*\mu}} - {\rm i}\, \partial_\mu \, \frac{\hat{\partial} g_3}{\hat{\partial} R_a^*} \,. \tag{B.24}$$

Eq. (B.23) implies $k^{i_{\rm f}}(c,\xi,\bar{\xi}) = 0$ by the same arguments as in the text after (B.22). (B.24) states that the 1-forms $dx^\mu H_{b\mu} k^{[ab]}(c,\xi,\bar{\xi})$ are trivial in the characteristic cohomology. This is equivalent to the statement that the 4-forms $d^4x R_b^* k^{[ab]}(c,\xi,\bar{\xi})$ are trivial in $H_3^4(\delta|d)$, see text before (B.18). Using (6.4), which we have already proved, we thus conclude from (B.24) that the $k^{[ab]}(c,\xi,\bar{\xi})$ vanish. □

Proof of lemma 7. We decompose the cocycle condition $s_{\rm ext}\omega^{g,4} + d\omega^{g-1,3} = 0$ into pieces with different antifield numbers. This gives

$$\delta\omega_{\underline{k}}^{g,4} + d\omega_{\underline{k}-1}^{g-1,3} = 0, \qquad \gamma_{\rm ext}\omega_{\underline{k}}^{g,4} + \delta\omega_{\underline{k}+1}^{g,4} + d\omega_{\underline{k}}^{g-1,3} = 0, \ \dots \tag{B.25}$$

where $\omega_k^{g,4}$ and $\omega_k^{g-1,3}$ are the pieces with antifield number k contained in $\omega^{g,4}$ and $\omega^{g-1,3}$, respectively, and $\underline{k}$ is the smallest antifield number which occurs in the decomposition of $\omega^{g,4}$. The first equation in (B.25) states that $\omega_{\underline{k}}^{g,4}$ is a cocycle of $H_{\underline{k}}^4(\delta|d)$. Without loss of generality we can assume that it is nontrivial in $H_{\underline{k}}^4(\delta|d)$ because otherwise we could remove it from $\omega^{g,4}$ by subtracting a coboundary of $H^{g,4}(s_{\rm ext}|d)$ from it (if $\omega_{\underline{k}}^{g,4} = \delta\omega_{\underline{k}+1}^{g-1,4} + d\omega_{\underline{k}}^{g,3}$, consider $\omega^{g,4} - s_{\rm ext}\omega_{\underline{k}+1}^{g-1,4} - d\omega_{\underline{k}}^{g,3}$). Now, if $g < -3$, then $\underline{k} > 3$ (the antifield number of a form cannot be smaller than minus its ghost number). Since $H_{\underline{k}}^4(\delta|d)$ vanishes for $\underline{k} > 3$ according to lemma 6, we conclude that $H^{g,4}(s_{\rm ext}|d)$ vanishes for $g < -3$. This yields the result for $g < -3$ in (6.6). If $g = -3$, then $\underline{k} \geq 3$ and lemma 6 implies that we can assume that $\underline{k} = 3$ and that $\omega_3^{-3,4}$ is a linear combination of the 4-forms $R_a^* d^4x$ with numerical coefficients (constant ghosts cannot occur in these coefficients for $g = -3$ because R_a^* has ghost number -3). Since $R_a^* d^4x$ is a cocycle not only of $H(\delta|d)$ but also of $H(s_{\rm ext}|d)$ (owing to $s_{\rm ext} R_a^* = (\delta + \hat{c}^\mu\partial_\mu) R_a^* = \partial_\mu({\rm i}Q_a^{*\mu} + \hat{c}^\mu R_a^*)$), this yields the result for $g = -3$ in (6.6). If $g = -2$, then $\underline{k} \geq 2$. Now lemma 6 leaves two possibilities, $\underline{k} = 2$ or

$\underline{k} = 3$. In the case $\underline{k} = 2$ it implies that $\omega_2^{-2,4}$ is a linear combination of the 4-forms $C^*_{i_f} d^4x$ and $f_{ab} d^4x$ with numerical coefficients (as $C^*_{i_f}$ and f_{ab} have ghost number -2). One may now verify by direct computation that $f'_{ab} d^4x$ completes $f_{ab} d^4x$ to a cocycle of $H(s_{\rm ext}|d)$. $C^*_{i_f} d^4x$ is already a cocycle of $H(s_{\rm ext}|d)$ (owing to $s_{\rm ext} C^*_{i_f} = (\delta + \hat{c}^\mu \partial_\mu) C^*_{i_f} = \partial_\mu(-A^{*\mu}_{i_f} + \hat{c}^\mu C^*_{i_f})$). In the case $g = -2$, $\underline{k} = 3$ we conclude from lemma 6 that $\omega_3^{-2,4}$ is a linear combination of the 4-forms $R^*_a d^4x$ with coefficients $k^a(c, \xi, \bar{\xi})$ that are linear combinations of the constant ghosts. Owing to $\gamma_{\rm ext} R^*_a = \partial_\mu(\hat{c}^\mu R^*_a)$, the second equation in (B.25) imposes $[\gamma_{\rm ext} k^a(c, \xi, \bar{\xi})] R^*_a d^4x = \delta(\cdots) + d(\cdots)$, i.e., the triviality of $[\gamma_{\rm ext} k^a(c, \xi, \bar{\xi})] R^*_a d^4x$ in $H(\delta|d)$. This implies $\gamma_{\rm ext} k^a(c, \xi, \bar{\xi}) = 0$ according to (6.4) because $\gamma_{\rm ext} k^a(c, \xi, \bar{\xi})$ is again a function of the constant ghosts. $\gamma_{\rm ext} k^a(c, \xi, \bar{\xi}) = 0$ implies $k^a(c, \xi, \bar{\xi}) = 0$ since no linear combination of the constant ghosts is $\gamma_{\rm ext}$-closed as can be easily verified directly [or deduced from the Lie algebra cohomology of the Lorentz group, as $\gamma_{\rm ext}$ contains the Lorentz transformations; see also section 6.4]. This completes the proof of (6.6).

To prove (6.7) we use that $k^a R^*_a d^4x \sim 0$ is equivalent to $k^a R^*_a = s_{\rm ext} f + \partial_\mu f^\mu$ (for some f and f^μ) and take the Euler-Lagrange derivative of the latter equation with respect to R^*_a. This gives

$$k^a = -s_{\rm ext} \frac{\hat{\partial} f}{\hat{\partial} R^*_a} + \hat{c}^\mu \partial_\mu \frac{\hat{\partial} f}{\hat{\partial} R^*_a} - 2\xi \sigma^\mu \bar{\xi} \frac{\hat{\partial} f}{\hat{\partial} Q^{*\mu}_a} . \tag{B.26}$$

All terms on the right hand side depend on fields, antifields or constant ghosts while the left hand side is a pure number. Hence both sides of (B.26) must vanish and we conclude $k^a = 0$. (6.8) can be proved analogously by taking the Euler-Lagrange derivatives of the equation $k^{i_f} C^*_{i_f} + k^{[ab]} f'_{ab} = s_{\rm ext} f + \partial_\mu f^\mu$ with respect to $C^*_{i_f}$ and $Q^{*\mu}_a$, respectively. This gives

$$k^{i_f} = s_{\rm ext} \frac{\hat{\partial} f}{\hat{\partial} C^*_{i_f}} - \hat{c}^\mu \partial_\mu \frac{\hat{\partial} f}{\hat{\partial} C^*_{i_f}} + 2{\rm i}\, \xi \sigma^\mu \bar{\xi} \frac{\hat{\partial} f}{\hat{\partial} A^{*\mu}_{i_f}} \tag{B.27}$$

$$k^{[ab]} H_{\mu b} = s_{\rm ext} \frac{\hat{\partial} f}{\hat{\partial} Q^{*\mu}_a} - \hat{c}^\nu \partial_\nu \frac{\hat{\partial} f}{\hat{\partial} Q^{*\mu}_a} + c_\mu{}^\nu \frac{\hat{\partial} f}{\hat{\partial} Q^{*\nu}_a} - {\rm i}\, \partial_\mu \frac{\hat{\partial} f}{\hat{\partial} R^*_a} + 2{\rm i}\, \xi \sigma^\nu \bar{\xi} \frac{\hat{\partial} f}{\hat{\partial} B^{*\mu\nu}_a} . \tag{B.28}$$

Eq. (B.27) implies $k^{i_f} = 0$ by the same arguments as the text after (B.26). Now consider (B.28). The ghost-independent part of that equation is of the form $k^{[ab]} H_{\mu b} = \delta(\cdots) + \partial_\mu(\cdots)$ which implies $k^{[ab]} = 0$ by the same arguments as in the text after (B.24). $\square$

Proof of lemma 8. We use that $H^g(s_{\rm ext}, \mathfrak{F})$ is isomorphic to $H^{g-4,4}(s_{\rm ext}|d) \oplus H^g(s_{\rm ext}, \mathfrak{E})$, where the representatives of $H^{g-4,4}(s_{\rm ext}|d)$ and $H^g(s_{\rm ext}, \mathfrak{F})$ are related through descent equations for $s_{\rm ext}$ and d, see lemma 1 and its proof. According to lemma 7, $H^{g-4,4}(s_{\rm ext}|d)$ is represented for $g < 3$ by linear combinations of $R^*_a d^4x$, $C^*_i d^4x$ and $f'_{ab} d^4x$. These 4-forms are related through descent equations for $s_{\rm ext}$ and d to the 0-forms ${\rm i}\tilde{H}^b \delta_{ab}$, $\tilde{F}^{j_f} \delta_{i_f j_f}$ and $(1/2)\tilde{H}^c \tilde{H}^d \delta_{ca} \delta_{db}$, respectively, as can be verified by direct computation. The contributions of $H(s_{\rm ext}|d)$ to the cohomology groups $H^g(s_{\rm ext}, \mathfrak{F})$, $g < 3$, are thus representatives $k_a \tilde{H}^a$ (for $g = 1$), and $k_{i_f} \tilde{F}^{i_f} + (1/2) k_{[ab]} \tilde{H}^a \tilde{H}^b$ (for $g = 2$). The contribution of $H(s_{\rm ext}, \mathfrak{E})$ is exhausted by complex numbers for $g = 0$. The reason is that $s_{\rm ext}$ contains the Lorentz transformations which enforces that representatives of $H(s_{\rm ext}, \mathfrak{E})$ be Lorentz-invariant (as the Lie algebra cohomology of the Lorentz-algebra enters here, see section 6.4 for details); however it is impossible to build nonvanishing Lorentz-invariants with ghost number 1 or 2 in $\mathfrak{E}$ owing to the Grassmann gradings of the constant ghosts (in particular, $\hat{c}^\mu \hat{c}_\mu$ and $\xi^\alpha \xi_\alpha$ vanish). This leaves us with $H^0(s_{\rm ext}, \mathfrak{E})$ which is evidently represented just by complex numbers because functions in $\mathfrak{E}$ with ghost number 0 are pure numbers. This ends the proof since $\mathfrak{E}$ does not contain polynomials with negative ghost numbers. $\square$

Proof of lemma 9. The proof is standard and uses a contracting homotopy ϱ for the u's and v's. ϱ is defined in the space of functions $f(u, v, w)$ according to

$$\varrho f(u,v,w) = \int_0^1 \frac{dt}{t}\, u^\ell\, \frac{\partial f(tu,tv,w)}{\partial v^\ell}.$$

Owing to (4.1), one has

$$\{s_{\rm ext}, \varrho\} f(u,v,w) = \int_0^1 \frac{dt}{t} \Big[u^\ell \frac{\partial}{\partial u^\ell} + v^\ell \frac{\partial}{\partial v^\ell}\Big] f(tu,tv,w) = f(u,v,w) - f(0,0,w)\,.$$

This implies

$$s_{\rm ext} f(u,v,w) = 0 \;\Rightarrow\; f(u,v,w) = f(0,0,w) + s_{\rm ext}\varrho f(u,v,w)\,,$$

i.e., only the piece $f(0,0,w)$ contained in an $s_{\rm ext}$-cocycle $f(u,v,w)$ can be nontrivial. Owing to $f(0,0,w) \in \mathfrak{W}$ and $s_{\rm ext}\mathfrak{W} \subset \mathfrak{W}$ (the latter follows from (4.1)), one concludes $H(s_{\rm ext}, \mathfrak{F}) \simeq H(s_{\rm ext}, \mathfrak{W})$. $\square$

Proof of lemma 10. Since coboundaries of $s_{\rm lie}$ can be removed from $f_{\overline{m}}$ (see text at the beginning of section 6.4), we conclude from (6.25) that we can assume $f_{\overline{m}} = f^{\Gamma_{\overline{m}}} P_{\Gamma_{\overline{m}}}(\theta, \hat{R})$ for some $f^{\Gamma_{\overline{m}}} \in \mathfrak{T}_{\rm inv}$. Using this in eq. (6.18) we obtain $(s_{\rm susy} f^{\Gamma_{\overline{m}}}) P_{\Gamma_{\overline{m}}}(\theta, \hat{R}) + s_{\rm lie} f_{\overline{m}-1} = 0$. The latter implies $s_{\rm susy} f^{\Gamma_{\overline{m}}} = 0$ for all $f^{\Gamma_{\overline{m}}}$ because of (6.26) (since $f^{\Gamma_{\overline{m}}} \in \mathfrak{T}_{\rm inv}$ implies $s_{\rm susy} f^{\Gamma_{\overline{m}}} \in \mathfrak{T}_{\rm inv}$). Assume now that $f^{\Gamma_{\overline{m}}} = s_{\rm susy} g$ for one of the $f^{\Gamma_{\overline{m}}}$ and some $g \in \mathfrak{T}_{\rm inv}$. Then this $f^{\Gamma_{\overline{m}}}$ can be removed from $f_{\overline{m}}$ by the redefinition $f - s_{\rm ext}[g P_{\Gamma_{\overline{m}}}(\theta, \hat{R})]$ (this redefinition only removes the piece with $f^{\Gamma_{\overline{m}}}$ from $f_{\overline{m}}$ and modifies in addition $f_{\overline{m}-1}$). Hence we can indeed assume without loss of generality that $f^{\Gamma_{\overline{m}}} \neq s_{\rm susy} g^{\Gamma_{\overline{m}}}$ for all $f^{\Gamma_{\overline{m}}}$. $\square$

Proof of lemma 11. The proof is based on methods and results developed in [61] and [12]. Therefore we shall only sketch the main line of reasoning and refer for details to the respective sections and equations in these works.

(i) The results (6.30) for $g = 0, 1, 2$ arise as follows. Owing to (6.28), nontrivial representatives of $H^g(s_{\rm ext}, \mathfrak{F})$ which are in $\mathfrak{T}_{\rm inv}$ are also nontrivial representatives of $H^g(s_{\rm susy}, \mathfrak{T}_{\rm inv})$. For $g = 0, 1, 2$ these are given by lemma 8. In addition $H^g(s_{\rm susy}, \mathfrak{T}_{\rm inv})$ contains functions $f \in \mathfrak{T}_{\rm inv}$ which are trivial in $H^g(s_{\rm ext}, \mathfrak{F})$ but nontrivial in $H^g(s_{\rm susy}, \mathfrak{T}_{\rm inv})$: $f = s_{\rm ext}\beta$ for some β, but not for any $\beta \in \mathfrak{T}_{\rm inv}$. For $g < 3$, the latter are exhausted by (linear combinations of) the abelian $\mathcal{F}$'s, up to trivial terms:

$$f = s_{\rm ext}\beta,\ f \in \mathfrak{T}_{\rm inv},\ \beta \in \mathfrak{F} \qquad \Rightarrow \qquad f = \begin{cases} 0 & \text{if} \quad g = 0 \\ s_{\rm ext}\beta',\ \beta' \in \mathfrak{T}_{\rm inv} & \text{if} \quad g = 1 \\ k_{i_A}\mathcal{F}^{i_A} + s_{\rm ext}\beta',\ \beta' \in \mathfrak{T}_{\rm inv} & \text{if} \quad g = 2. \end{cases} \tag{B.29}$$

This can be proved as an analogous result in appendix D of [12]. The basic ideas are as follows. One can assume $\beta \in \mathfrak{W}$, see lemma 9. The result (B.29) for $g = 0$ simply reflects that $\mathfrak{W}$ contains no function with negative ghost number, i.e., there is no β with ghost number -1. The result (B.29) for $g = 1$ holds because functions in $\mathfrak{W}$ with ghost number 0 can only depend on the $\hat{T}$'s which implies $\beta = \beta(\hat{T}) \in \mathfrak{T}_{\rm inv}$. The result (B.29) for $g = 2$ is obtained by differentiation of $f = s_{\rm ext}\beta$ with respect to $\hat{C}^i$ or $c^{\mu\nu}$ which yields $s_{\rm ext}(\partial\beta/\partial\hat{C}^i) = 0$ and $s_{\rm ext}(\partial\beta/\partial c^{\mu\nu}) = 0$, without loss of generality (see [12]). Using now lemma 8 and that $\partial\beta/\partial\hat{C}^i$ and $\partial\beta/\partial c^{\mu\nu}$ have ghost number 0 in the case $g = 2$, one concludes that these expressions are complex numbers, and thus $\beta = k'_i\hat{C}^i + k'_{\mu\nu}c^{\mu\nu} + \beta'(\hat{c},\xi,\bar{\xi},\hat{T})$. $f = s_{\rm ext}\beta \in \mathfrak{T}_{\rm inv}$ implies then $k'_i = 0$ unless δ_i is abelian, $k'_{\mu\nu} = 0$, and $\beta' \in \mathfrak{T}_{\rm inv}$.

The results (6.30) for $g \geq 3$ are derived using a decomposition of the cocycle condition $s_{\rm susy}f = 0$ with respect to the degree in the translation ghosts $\hat{c}^\mu$ ("$\hat{c}$-degree"). $s_{\rm susy}$ decomposes into pieces δ_-, δ_0, δ_+ with $\hat{c}$-degrees -1, 0, 1, respectively, where δ_0 consists of two differentials b and $\bar{b}$ that involve ξ and $\bar{\xi}$, respectively:

$$\begin{aligned} &s_{\rm susy} = \delta_- + \delta_0 + \delta_+\,, \qquad \delta_0 = b + \bar{b}\,, \\ &\delta_- = 2\mathrm{i}\xi\sigma^\mu\bar{\xi}\,\frac{\partial}{\partial\hat{c}^\mu}\,, \qquad b = \xi^\alpha\mathcal{D}_\alpha\,, \qquad \bar{b} = \bar{\xi}^{\dot{\alpha}}\bar{\mathcal{D}}_{\dot{\alpha}}\,, \qquad \delta_+ = \hat{c}^\mu\hat{\nabla}_\mu\,. \end{aligned} \tag{B.30}$$

Here $\mathcal{D}_\alpha$, $\bar{\mathcal{D}}_{\dot{\alpha}}$ and $\hat{\nabla}_\mu$ act nontrivially only on the $\hat{T}$'s. To prove the results (6.30) for $g \geq 3$ one only needs the cohomology of δ_-, and the cohomologies of b and $\bar{b}$ in certain spaces specified below. The cohomology of δ_- was given in eq. (6.5) of [61] (see also [64]); it reads:

$$\begin{aligned} \delta_- f(\hat{c},\xi,\bar{\xi}) = 0 \ \Leftrightarrow\ & f(\hat{c},\xi,\bar{\xi}) = P(\bar{\vartheta},\xi) + Q(\vartheta,\bar{\xi}) + \Theta M + \delta_- g(\hat{c},\xi,\bar{\xi})\,, \\ & \Theta = \hat{c}^\mu\xi\sigma_\mu\bar{\xi}, \qquad \vartheta^\alpha = \hat{c}^\mu(\bar{\xi}\bar{\sigma}_\mu)^\alpha, \qquad \bar{\vartheta}^{\dot{\alpha}} = \hat{c}^\mu(\bar{\sigma}_\mu\xi)^{\dot{\alpha}}\,, \end{aligned} \tag{B.31}$$

where nonvanishing $P(\bar{\vartheta},\xi)$, $Q(\vartheta,\bar{\xi})$ and ΘM are nontrivial in $H(\delta_-)$. This result holds in the space of all polynomials in the $\hat{c}^\mu$, ξ^α, $\bar{\xi}^{\dot{\alpha}}$, irrespectively of whether or not they are Lorentz-invariant. Of course, it holds analogously in the space of polynomials that can also depend on the $\hat{T}$'s because the latter are inert to δ_- (in that space (B.31) thus holds with f, P, Q, M, g depending also on the $\hat{T}$'s). The cohomology of $\bar{b}$ is needed in the space of local functions of the $\bar{\xi}^{\dot{\alpha}}$ and $\hat{T}^\tau$ which are invariant under $\mathfrak{sl}(2,\mathbb{C})$-transformations of the dotted spinor indices.[22] This cohomology is given by

$$\begin{aligned} \bar{b}f(\bar{\xi},\hat{T}) = \bar{l}_{\dot{\alpha}\dot{\beta}}f(\bar{\xi},\hat{T}) = 0 \qquad \Leftrightarrow \qquad & f(\bar{\xi},\hat{T}) = A(\varphi,\hat{\lambda}) + \bar{\mathcal{D}}^2B(\hat{T}) + \bar{b}g(\bar{\xi},\hat{T}), \\ & \bar{l}_{\dot{\alpha}\dot{\beta}}B(\hat{T}) = \bar{l}_{\dot{\alpha}\dot{\beta}}g(\bar{\xi},\hat{T}) = 0\,, \end{aligned} \tag{B.32}$$

where $\bar{l}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}\bar{\sigma}^{\mu\nu}_{\dot{\alpha}\dot{\beta}}l_{\mu\nu}$ generates the $\mathfrak{sl}(2,\mathbb{C})$-transformations of the dotted spinor indices ($\bar{l}_{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\gamma}} = -\varepsilon_{\dot{\gamma}(\dot{\alpha}}\bar{\psi}_{\dot{\beta})}$ etc.). (B.32) can be derived from its linearized version which concerns the cohomology of the operator $\bar{b}_0 = \bar{\xi}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}}$ involving only the linear part $\bar{D}_{\dot{\alpha}}$ of $\bar{\mathcal{D}}_{\dot{\alpha}}$ (i.e., $\bar{D}_{\dot{\alpha}}\hat{T}^\tau$ is the part of $\bar{\mathcal{D}}_{\dot{\alpha}}\hat{T}^\tau$ which is linear in the $\hat{T}$'s; e.g., one has $\bar{\mathcal{D}}_{\dot{\alpha}}\hat{\bar{\lambda}}^i_{\dot{\beta}} = -\hat{F}^{i(-)}_{\dot{\alpha}\dot{\beta}} +$

[22] $\bar{b}$ is indeed a differential because of $\bar{b}^2 = (1/2)\bar{\xi}^{\dot{\alpha}}\bar{\xi}^{\dot{\beta}}\{\bar{\mathcal{D}}_{\dot{\alpha}},\bar{\mathcal{D}}_{\dot{\beta}}\} = 0$ which follows from (5.9).

$\varepsilon_{\dot\alpha\dot\beta}\delta^{ij}\bar\varphi T_j\varphi$ and thus $\bar D_{\dot\alpha}\bar{\hat\lambda}{}^i_{\dot\beta}=-\hat F^{i(-)}_{\dot\alpha\dot\beta}$) and is given by:

$$\bar b_0 f(\bar\xi,\hat T)=\bar l_{\dot\alpha\dot\beta}f(\bar\xi,\hat T)=0 \qquad\Leftrightarrow\qquad \begin{aligned}&f(\bar\xi,\hat T)=A(\varphi,\hat\lambda)+\bar D^2B(\hat T)+\bar b_0 g(\bar\xi,\hat T)\,,\\ &\bar l_{\dot\alpha\dot\beta}B(\hat T)=\bar l_{\dot\alpha\dot\beta}g(\bar\xi,\hat T)=0\,.\end{aligned} \tag{B.33}$$

This follows from eq. (6.8) of [61] because the $\bar D_{\dot\alpha}$-representation on the $\hat T^\tau$ has QDS structure in the terminology used there: indeed, the $\bar D_{\dot\alpha}$-representation decomposes into the singlets φ^s and $\hat\lambda^i$ and infinitely many (D)-doublets given by

$$\begin{array}{ll}((\hat\nabla^+_+)^q\bar{\hat\lambda}{}^i,-(\hat\nabla^+_+)^q\bar{\hat F}{}^{i(-)})\,, & ((\hat\nabla^+_+)^q\hat F^{i(+)},2\mathrm{i}(\hat\nabla^+_+)^{q+1}\hat\lambda^i)\,,\\ ((\hat\nabla^+_+)^q\phi^a,(\hat\nabla^+_+)^q\bar{\hat\psi}{}^a)\,, & ((\hat\nabla^+_+)^q\hat\psi^a,(\hat\nabla^+_+)^q\hat H^a-\mathrm{i}(\hat\nabla^+_+)^{q+1}\phi^a)\,,\\ ((\hat\nabla^+_+)^q\bar\varphi^s,(\hat\nabla^+_+)^q\bar{\hat\chi}{}^s)\,, & ((\hat\nabla^+_+)^q\hat\chi^s,-2\mathrm{i}(\hat\nabla^+_+)^{q+1}\varphi^s)\end{array}$$

(where $q=0,1,\dots$). (B.32) follows straightforwardly from (B.33) by a standard argument: let f be a $\bar b$-cocycle; one can decompose f according to $f=\sum_{k\geq k_0}f_k$ into parts f_k with definite degree k in the $\hat T$'s where k_0 denotes the lowest nonvanishing degree of the decomposition; $\bar bf=0$ implies $\bar b_0 f_{k_0}=0$; (B.33) implies $f_{k_0}=A_{k_0}(\varphi,\hat\lambda)+\bar D^2B_{k_0}(\hat T)+\bar b_0 g_{k_0}(\bar\xi,\hat T)$ for some $A_{k_0}(\varphi,\hat\lambda)$, $B_{k_0}(\hat T)$ and $g_{k_0}(\bar\xi,\hat T)$ that are $\bar l_{\dot\alpha\dot\beta}$-invariant; one now considers $f'=f-A_{k_0}(\varphi,\hat\lambda)-\bar{\mathcal D}^2B_{k_0}(\hat T)-\bar bg_{k_0}(\bar\xi,\hat T)$; by construction f' is $\bar b$-invariant and its decomposition starts at some degree $k_0'>k_0$; one repeats the arguments for f' and goes on until one has proved (B.32) with $A=A_{k_0}+A_{k_0'}+\cdots$, $B=B_{k_0}+B_{k_0'}+\cdots$ and $g=g_{k_0}+g_{k_0'}+\cdots$

Armed with (B.31) and (B.32), one can prove the results (6.30) for $g\geq 4$ as in section 6 of [61] (the arguments used in [61] go through in $\mathfrak T_{\rm inv}$ because all relevant operations used there commute with the δ_i) and for $g=3$ as the corresponding result in appendix E of [12].

(ii) The nontriviality and inequivalence of the cocycles 1, $\tilde H^a$, $\tilde F^{i_{\rm f}}$ and $\tilde H^a\tilde H^b$, $a<b$, in $H(s_{\rm susy},\mathfrak T_{\rm inv})$ follows already from (6.10), (6.11) and (6.12). The nontriviality and inequivalence of the $\mathcal F^{i_A}$ in $H(s_{\rm susy},\mathfrak T_{\rm inv})$ can be shown as the corresponding result in appendix E of [12]. That $\mathcal F^{i_A}$ is not equivalent to $\tilde F^{i_{\rm f}}$ or $\tilde H^a\tilde H^b$ can be deduced from its $s_{\rm ext}$-exactness, $\mathcal F^{i_A}=s_{\rm ext}\hat C^{i_A}$: $k_{i_{\rm f}}\tilde F^{i_{\rm f}}+\frac12 k_{ab}\tilde H^a\tilde H^b+k_{i_A}\mathcal F^{i_A}=s_{\rm susy}\omega$ for some $\omega\in\mathfrak T_{\rm inv}$ implies $k_{i_{\rm f}}\tilde F^{i_{\rm f}}+\frac12 k_{ab}\tilde H^a\tilde H^b=s_{\rm ext}(\omega-k_{i_A}\hat C^{i_A})$ and thus $k_{i_{\rm f}}=k_{[ab]}=0$ by (6.12). This proves the assertion on the cocycles 1, $\tilde H^a$, $\tilde F^{i_{\rm f}}$, $\tilde H^a\tilde H^b$ and $\mathcal F^{i_A}$ in part (ii).

Next we shall prove the assertion on $\mathcal{O}R(\hat T)$. Assume that $\mathcal{O}R(\hat T)=s_{\rm susy}\omega$ for some $\omega\in\mathfrak T_{\rm inv}$. Notice that ω is defined only up to the addition of $s_{\rm susy}$-cocycles in $\mathfrak T_{\rm inv}$. In particular we are free to add terms $s_{\rm susy}f$ with $f\in\mathfrak T_{\rm inv}$ to ω (owing to $s_{\rm susy}^2f=0$ for $f\in\mathfrak T_{\rm inv}$). Since ω has ghost number 2 in this case, it decomposes into $\omega_0+\omega_1+\omega_2$ where subscripts indicate the $\hat c$-degree. One has $\omega_0=\xi^\alpha\xi^\beta\omega_{\alpha\beta}(\hat T)+\xi^\alpha\bar\xi^{\dot\alpha}\omega_{\alpha\dot\alpha}(\hat T)+\bar\xi^{\dot\alpha}\bar\xi^{\dot\beta}\omega_{\dot\alpha\dot\beta}(\hat T)$. The part of $\mathcal{O}R(\hat T)=s_{\rm susy}\omega$ with $\hat c$-degree 0 reads $0=\delta_0\omega_0+\delta_-\omega_1$. This gives in particular $\bar b[\bar\xi^{\dot\alpha}\bar\xi^{\dot\beta}\omega_{\dot\alpha\dot\beta}(\hat T)]=0$. (B.32) implies thus $\bar\xi^{\dot\alpha}\bar\xi^{\dot\beta}\omega_{\dot\alpha\dot\beta}(\hat T)=\bar b\eta$ for some $\eta=\bar\xi^{\dot\alpha}\eta_{\dot\alpha}(\hat T)\in\mathfrak T_{\rm inv}$. Without loss of generality one can thus set $\omega_{\dot\alpha\dot\beta}(\hat T)$ to zero because the latter can be removed by subtracting $s_{\rm susy}\eta$ from ω. Analogously one concludes that one can also set $\omega_{\alpha\beta}(\hat T)$ to zero. One is then left with $\omega_0=\xi^\alpha\bar\xi^{\dot\alpha}\omega_{\alpha\dot\alpha}(\hat T)$ which can also be set to zero because it can be removed from ω by adding $s_{\rm susy}a$ with $a=(\mathrm{i}/4)\hat c^{\alpha\dot\alpha}\omega_{\alpha\dot\alpha}(\hat T)$. Hence, one can assume

$\omega = \omega_1 + \omega_2$ with $\delta_-\omega_1 = 0$. Using (B.31) and that ω has ghost number 2, one concludes $\omega_1 = 8\vartheta\Omega_1 + 8\bar{\vartheta}\bar{\Omega}_2$ for some functions Ω_1^α and $\bar{\Omega}_2^{\dot{\alpha}}$ of the $\hat{T}$'s that are gauge invariant and transform under Lorentz transformations as indicated by their indices [ω_1 cannot contain a δ_--exact piece because it has ghost number 2 and $\hat{c}$-degree 1 and thus depends linearly on the supersymmetry ghosts whereas δ_--exact terms depend at least quadratically on them]. The part of $\mathcal{O}R(\hat{T}) = s_{\mathrm{susy}}\omega$ with $\hat{c}$-degree 1 reads $4\Theta R(\hat{T}) = \delta_0\omega_1 + \delta_-\omega_2$. This gives in particular $\bar{b}\Omega_1^\alpha = 0$. Using (B.32) one concludes $\Omega_1^\alpha = A_1^\alpha(\varphi, \hat{\lambda}) + \bar{\mathcal{D}}^2 B_1^\alpha(\hat{T})$ for some functions $A_1^\alpha(\varphi, \hat{\lambda})$ and $B_1^\alpha(\hat{T})$. Analogously one concludes $\bar{\Omega}_2^{\dot{\alpha}}(\mathcal{T}) = \bar{A}_2^{\dot{\alpha}}(\bar{\varphi}, \hat{\lambda}) + \mathcal{D}^2\bar{B}_2^{\dot{\alpha}}(\hat{T})$. Using this in $4\Theta R(\hat{T}) = \delta_0\omega_1 + \delta_-\omega_2$ one obtains $4\Theta R(\hat{T}) + 8\vartheta b\Omega_1 + 8\bar{\vartheta}\bar{b}\bar{\Omega}_2 = \delta_-\omega_2$. By means of the δ_--cohomology one concludes $R(\hat{T}) = \mathcal{D}\Omega_1 + \bar{\mathcal{D}}\bar{\Omega}_2$ and $\omega_2 = -\mathrm{i}\hat{c}_{\alpha\dot{\alpha}}\hat{c}^{\dot{\alpha}}{}_\beta\mathcal{D}^\alpha\Omega_1^\beta - \mathrm{i}\hat{c}^\alpha{}_{\dot{\alpha}}\hat{c}_{\alpha\dot{\beta}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Omega}_2^{\dot{\beta}}$ [ω_2 cannot contain a δ_--exact piece because it has ghost number 2 and $\hat{c}$-degree 2 and thus does not depend on the supersymmetry ghosts]. Conversely, $R(\hat{T}) = \mathcal{D}\Omega_1 + \bar{\mathcal{D}}\bar{\Omega}_2$ with Ω_1^α and $\bar{\Omega}_2^{\dot{\alpha}}$ as above implies $\mathcal{O}R(\hat{T}) = s_{\mathrm{susy}}(\omega_1 + \omega_2)$ with ω_1 and ω_2 as above.

The assertion on $\mathcal{P}\Omega_1 + \bar{\mathcal{P}}\bar{\Omega}_2$ is proved in an analogous manner by decomposing the equation $\mathcal{P}\Omega_1 + \bar{\mathcal{P}}\bar{\Omega}_2 = s_{\mathrm{susy}}\omega$ into parts with different $\hat{c}$-degrees. The parts with $\hat{c}$-degree 0 and 1 read $0 = \delta_0\omega_0 + \delta_-\omega_1$ and $0 = \delta_+\omega_0 + \delta_0\omega_1 + \delta_-\omega_2$, respectively. As in the investigation of the case $G = 3$ in [12] one concludes from these equations that, up to trivial terms, one has $\omega_0 = 0$, $\omega_1 = 32\Theta X(\mathcal{T})$, $\omega_2 = 8\mathrm{i}\hat{c}^{\dot{\alpha}\alpha}(\bar{\vartheta}_{\dot{\alpha}}\mathcal{D}_\alpha + \vartheta_\alpha\bar{\mathcal{D}}_{\dot{\alpha}})X$ for some $X = X(\hat{T}) \in \mathfrak{T}_{\mathrm{inv}}$. Using this in the equation with $\hat{c}$-degree 2 contained in $\mathcal{P}\Omega_1 + \bar{\mathcal{P}}\bar{\Omega}_2 = s_{\mathrm{susy}}\omega$, one obtains $4\mathrm{i}\vartheta\vartheta(\Omega_1 - \bar{\mathcal{D}}^2X) - 4\mathrm{i}\bar{\vartheta}\bar{\vartheta}(\bar{\Omega}_2 + \mathcal{D}^2X) = -8\mathrm{i}\vartheta_\alpha\bar{\vartheta}_{\dot{\alpha}}[\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}]X + \delta_-\omega_3$. By means of the δ_--cohomology one concludes $\Omega_1 = \bar{\mathcal{D}}^2X$, $\bar{\Omega}_2 = -\mathcal{D}^2X$ and $\omega_3 = -4\mathrm{i}\,\Xi_{\alpha\dot{\alpha}}[\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}]X$. Furthermore, using the algebra (5.9), one verifies that $s_{\mathrm{ext}}(\omega_1 + \omega_2 + \omega_3) = (\mathcal{P}\bar{\mathcal{D}}^2 - \bar{\mathcal{P}}\mathcal{D}^2)X$ and concludes that $\mathcal{P}\Omega_1 + \bar{\mathcal{P}}\bar{\Omega}_2 = s_{\mathrm{susy}}\omega$ is indeed equivalent to $\Omega_1 = \bar{\mathcal{D}}^2X$, $\bar{\Omega}_2 = -\mathcal{D}^2X$ (I note that one has $\omega_1 + \omega_2 + \omega_3 = 8\mathcal{O}X$). □

Proof of lemma 12. (i) $s_{\mathrm{ext}}f_4 = 0$ implies $\delta_-f_4 = 0$ because δ_- is the only part of s_{ext} that lowers the $\hat{c}$-degree. (B.31) implies that the cohomology of δ_- is trivial for $\hat{c}$-degrees larger than 2 because ϑ^α and $\bar{\vartheta}^{\dot{\alpha}}$ are anticommuting quantities. We conclude that f_4 is δ_--exact, i.e., $f_4 = \delta_-\eta_5$. This implies $f_4 = 0$ because η_5 has $\hat{c}$-degree 5 and thus vanishes.

(ii) Existence of η_4: $s_{\mathrm{ext}}(f_3 + f_4) = 0$ implies $\delta_-f_3 = 0$, from which we conclude $f_3 = \delta_-\eta_4$ by means of (B.31). Consider now $f' := f_3 + f_4 - s_{\mathrm{ext}}\eta_4$: it is an s_{ext}-closed function with $\hat{c}$-degree 4. By means of part (i) we conclude that f' vanishes and thus that $f_3 + f_4 = s_{\mathrm{ext}}\eta_4$. Uniqueness of η_4: $f_3 + f_4 = s_{\mathrm{ext}}\eta_4$ and $f_3 + f_4 = s_{\mathrm{ext}}\eta'_4$ imply $s_{\mathrm{ext}}(\eta_4 - \eta'_4) = 0$ and thus $\eta_4 - \eta'_4 = 0$ according to part (i). □

Proof of lemma 13. We first recall the well-known Chern-Simons-polynomials $q_r(A, F) = m(r)\int_0^1 dt\ \mathrm{tr}_r(AF_t^{m(r)-1})$ with $F_t = tF + (t^2 - t)A^2$ for Yang-Mills connection forms A and the curvature forms $F = dA + A^2$. They satisfy $dq_r(A, F) = \mathrm{tr}_r\, F^{m(r)}$, see, e.g., [65]. Now, written in terms of matrices, eq. (5.1) reads $s_{\mathrm{ext}}\hat{C} = -\hat{C}^2 + \mathcal{F}$ which arises from $F = dA + A^2$ by substituting s_{ext}, $\hat{C}$ and $\mathcal{F}$ for d, A and F, respectively. The action of s_{ext} on polynomials in the $\hat{C}$ and $\mathcal{F}$ is thus isomorphic to the action of d on polynomials in the A and F (this statement refers to the free differential algebras of $\hat{C}$'s and $\mathcal{F}$'s, and

A's and F's, respectively). We conclude

$$s_{\rm ext} q_r(\hat{C},\mathcal{F}) = \mathrm{tr}_r\, \mathcal{F}^{m(r)}, \qquad q_r(\hat{C},\mathcal{F}) = m(r)\int_0^1 dt\ \mathrm{tr}_r\left(\hat{C}\mathcal{F}_t^{m(r)-1}\right), \tag{B.34}$$

for all $r = 1,\dots,\mathrm{rank}(\mathfrak{g}_{\rm YM})$. We shall now show that $\mathrm{tr}_r\,\mathcal{F}^{m(r)}$ vanishes for $m(r) > 3$. This follows from the explicit form of $\mathcal{F}$: eq. (6.14) gives

$$\mathcal{F} = \hat{F} - \mathrm{i}\vartheta\hat{\lambda} - \mathrm{i}\bar{\vartheta}\hat{\bar{\lambda}}\,, \qquad \hat{F} = \frac{1}{2}\hat{c}^\mu\hat{c}^\nu\hat{F}^i_{\mu\nu}T_i^{(r)}\,, \tag{B.35}$$

with ϑ^α and $\bar{\vartheta}^{\dot\alpha}$ as in (B.31). Since ϑ^α and $\bar{\vartheta}^{\dot\alpha}$ have $\hat{c}$-degree 1, $\mathrm{tr}_r\,\mathcal{F}^{m(r)}$ contains only terms with $\hat{c}$-degree $\geq m(r)$. In particular it thus vanishes for $m(r) > 4$ (since the spacetime dimension is 4 and the translation ghosts anticommute). For $m(r) = 4$ one obtains:

$$\mathrm{tr}_r\,\mathcal{F}^4 = \mathrm{tr}_r\left(\vartheta\hat{\lambda} + \bar{\vartheta}\hat{\bar{\lambda}}\right)^4. \tag{B.36}$$

Since the ϑ^α and $\bar{\vartheta}^{\dot\alpha}$ are four anticommuting quantities, (B.36) is proportional to the product of all these four quantities and thus to $\vartheta\vartheta\bar{\vartheta}\bar{\vartheta}$. The latter object has $\hat{c}$-degree 4, is bilinear both in the ξ's and the $\bar{\xi}$'s, and Lorentz-invariant; hence it is proportional to $\Xi\xi\xi\bar{\xi}\bar{\xi}$ which vanishes owing to $\xi\xi = \bar{\xi}\bar{\xi} = 0$ (as the supersymmetry ghosts commute). We conclude that (B.36) vanishes and thus

$$\mathrm{tr}_r\,\mathcal{F}^{m(r)} = 0 \qquad \text{for} \qquad m(r) \geq 4\,. \tag{B.37}$$

Eqs. (B.34) and (B.37) yield $s_{\rm ext}\hat{q}_r = 0$ for $m(r) \geq 4$.

For $m(r) = 3$, (B.34) gives explicitly:

$$s_{\rm ext}\left[\mathrm{tr}_r\left(\hat{C}\mathcal{F}^2 - \frac{1}{2}\hat{C}^3\mathcal{F} + \frac{1}{10}\hat{C}^5\right)\right] = \mathrm{tr}_r\,\mathcal{F}^3\,. \tag{B.38}$$

Evaluation of the right hand side gives, using (B.35) and the facts that ϑ^α and $\bar{\vartheta}^{\dot\alpha}$ anticommute and have $\hat{c}$-degree 1:

$$\begin{aligned} \mathrm{tr}_r\,\mathcal{F}^3 &= f_3 + f_4\,, \\ f_3 &= \mathrm{i}\,\mathrm{tr}_r(\vartheta\hat{\lambda} + \bar{\vartheta}\hat{\bar{\lambda}})^3 = -\frac{3\mathrm{i}}{2}\,\mathrm{tr}_r\left(\bar{\vartheta}\bar{\vartheta}\,\vartheta\hat{\lambda}\,\hat{\bar{\lambda}}\hat{\bar{\lambda}} + \vartheta\vartheta\,\bar{\vartheta}\hat{\bar{\lambda}}\,\hat{\lambda}\hat{\lambda}\right), \end{aligned} \tag{B.39}$$

where f_4 has $\hat{c}$-degree 4. Owing to $s_{\rm ext}(\mathrm{tr}_r\,\mathcal{F}^3) = 0$ (which follows from (B.38) because of $s_{\rm ext}^2 = 0$), we can apply part (ii) of lemma 12 to (B.39) and conclude:

$$\mathrm{tr}_r\,\mathcal{F}^3 = s_{\rm ext}\left[-3\mathrm{i}\Xi\,\mathrm{tr}_r\left(\xi\hat{\lambda}\hat{\bar{\lambda}}\hat{\bar{\lambda}} + \bar{\xi}\hat{\bar{\lambda}}\hat{\lambda}\hat{\lambda}\right)\right], \tag{B.40}$$

where we used that

$$\delta_-(2\Xi\xi^\alpha) = \bar{\vartheta}\bar{\vartheta}\vartheta^\alpha\,, \qquad \delta_-(2\Xi\bar{\xi}_{\dot\alpha}) = \vartheta\vartheta\bar{\vartheta}_{\dot\alpha} \tag{B.41}$$

which can be directly verified. Eqs. (B.38) and (B.40) yield $s_{\rm ext}\hat{q}_r = 0$ for $m(r) = 3$ with $\hat{q}_r$ as in (6.44). □

Proof of lemma 14. Let us first describe the general strategy of the proof. The derivation of the lemma starts off from our result that the part $f_{\overline{m}}$ of an $s_{\rm ext}$-cocycle takes the form $f^\Gamma P_\Gamma(\theta,\hat{R})$ where f^Γ are representatives of $H(s_{\rm susy},\mathfrak{T}_{\rm inv})$ given by lemma 11. The seven types of representatives given in the lemma correspond to different representatives f^Γ of $H(s_{\rm susy},\mathfrak{T}_{\rm inv})$. The proof of the lemma comprises two aspects: (i) determination of those functions $f^\Gamma P_\Gamma(\theta,\hat{R})$ which can be completed to inequivalent $s_{\rm ext}$-cocycles; (ii) explicit computation of these $s_{\rm ext}$-cocycles. These two aspects can be treated largely independently, i.e., basically one can carry out (i) without sophisticated computation. The computations (ii) concern above all the explicit determination of the X-functions given in the lemma and are partly quite involved. We shall only sketch the computation of the functions X^a_r. The other X's can be analogously derived.

(i) is carried out using results derived above, in particular lemma 11 (which implies already the existence of X-functions with the desired properties as we shall show in the course of the proof), and results on the standard (non-extended) BRST cohomology. The latter can be employed here because $s_{\rm ext}$ includes the standard (non-extended) BRST differential s (the s-transformations arise from the $s_{\rm ext}$-transformations by setting all constant ghosts c^μ, $c^{\mu\nu}$, ξ^α, $\bar{\xi}^{\dot{\alpha}}$ to zero). A necessary condition for a function $f\in\mathfrak{F}$ to be an $s_{\rm ext}$-cocycle is thus that it gives a solution to "complete" descent equations for s and d (where "complete" means descent equations involving a volume form) after setting $c^{\mu\nu}$, ξ^α and $\bar{\xi}^{\dot{\alpha}}$ to zero and substituting dx^μ for c^μ (the reason is that the substitution $c^\mu\to c^\mu+dx^\mu$ promotes an $s_{\rm ext}$-cocycle to an $(s_{\rm ext}+d)$-cocycle, i.e., to a solution of complete descent equations for $s_{\rm ext}$ and d, see section 3). In particular this implies that a function $f^\Gamma P_\Gamma(\theta,\hat{R})$ cannot be completed to an $s_{\rm ext}$-cocycle if there is no corresponding solution of complete descent equtions for s and d. Actually this statement can be refined because the relevant descent equations for s and d are those in the subspace of Poincaré invariant forms[23] with an arbitrary dependence on the Lorentz-θ's[24] (for the argument applies equally to an extended BRST-differential which involves the Poincaré transformations in addition to s and arises from $s_{\rm ext}$ by setting only the constant supersymmetry ghosts to zero).

We shall now spell out the arguments more specifically, and separately for the various types of representatives:

(1) The representatives $f^{(1)}$ arise from functions $k^\Gamma P_\Gamma(\theta,\hat{R})$ with complex numbers k^Γ, i.e., they involve the constant representatives of $H(s_{\rm susy},\mathfrak{T}_{\rm inv})$ ($g=0$ in eq. (6.30)). Functions $P(\theta,\hat{R})$ give rise to solutions of complete non-supersymmetric descent equations in four dimensions only if they do not depend on the abelian $\hat{C}$'s, nor on the Yang-Mills-θ's with $m(r)=2$, nor on the $\hat{R}$'s. The reason is that abelian ghosts, R's or Yang-Mills-θ's with $m(r)=2$ lead to obstructions to the "lift"[25]

[23] Poincaré invariance of a p-form $dx^{\mu_1}\cdots dx^{\mu_p}\omega_{[\mu_1\dots\mu_p]}$ means here that the coefficient functions $\omega_{[\mu_1\dots\mu_p]}$ do not depend explicitly on the spacetime coordinates and transform covariantly under Lorentz transformations according to their indices $[\mu_1\dots\mu_p]$.

[24] Since the Lorentz-θ's are $s_{\rm ext}$-closed and d-closed, they can appear arbitrarily in the solutions of the descent equations.

[25] See section 9.3 of [7] for the terminology and a general discussion of "lifts" in the context of descent equations.

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of 0-forms $P(\theta_C, \theta_{\rm L}, R)$ to solutions of complete descent equations where the θ_C's are the Yang-Mills-θ's with C's in place of $\hat{C}$'s (i.e., $(\theta_C)_r \propto \mathrm{tr}_r\, C^{2m(r)-1}$): if P depends on abelian ghosts, the obstruction is encountered at form-degree 2 and given by $F^{i_A}\partial P(\theta_C, \theta_{\rm L}, R)/\partial C^{i_A}$ where $F^{i_A} = dA^{i_A}$ with $A^{i_A} = dx^\mu A_\mu^{i_A}$; if P does not depend on abelian ghosts but on R's, the obstruction is encountered at form-degree 3 and given by $-\mathrm{i}H^a\partial P(\theta_C, \theta_{\rm L}, R)/\partial R^a$ where $H^a = dB^a$ with $B^a = (1/2)dx^\mu dx^\nu B^a_{\mu\nu}$; if P neither depends on abelian ghosts nor on R's but on Yang-Mills-θ's with $m(r) = 2$, the obstruction is encountered at form-degree 4 and given by $\sum_{r:m(r)=2} \mathrm{tr}_r\, F^2\, \partial P(\theta_C, \theta_{\rm L})/\partial(\theta_C)_r$ where $F = dA + A^2$, $A = dx^\mu A^i_\mu T^{(r)}_i$. Hence, in order that P can be lifted to a complete solution of the non-supersymmetric descent equations, it must only depend on the θ's with $m(r) > 2$ or the Lorentz-θ's. Since the former can be completed to $s_{\rm ext}$-invariant $\hat{q}$'s, see lemma 13, and the latter are already $s_{\rm ext}$-invariant, one arrives at the representatives $f^{(1)}$.

(2,3) The representatives $f^{(2)}$ and $f^{(3)}$ arise from functions $k^\Gamma_a \tilde{H}^a P_\Gamma(\theta, \hat{R})$ with complex numbers k^Γ_a, i.e., they involve the representatives $\tilde{H}^a$ of $H(s_{\rm susy}, \mathfrak{T}_{\rm inv})$. These functions give rise to Poincaré invariant solutions of the complete non-supersymmetric descent equations in four dimensions only if they do not depend on the abelian $\hat{C}$'s and at most linearly on the $\hat{R}$'s through terms $\tilde{H}^a P^{(2)}_a(\theta) + \tilde{H}^{[a}\hat{R}^{b]}P^{(3)}_{[ab]}(\theta)$. This can be shown by arguments analogous to those used in section 13 of [7] where the standard BRST cohomology for free abelian gauge fields was investigated. The latter leads, among other things, to the BRST-invariant forms $\star F^I P_I(C)$ where $F^I = dA^I$ are abelian field strength 2-forms, $\star F^I$ their Hodge duals and $P_I(C)$ polynomials in the abelian ghosts. The requirement that such forms can be lifted to Poincaré invariant solutions of complete descent equations leads to the equation $\partial_{(I}P_{J)}(C) = 0$ whose general solution is $P_I(C) = \partial_I P(C)$ where $\partial_I = \partial/\partial C^I$, see eqs. (13.18) and (13.19) of [7]. The standard BRST cohomology for the theories under study contains, among other things, the BRST invariant forms $\star H^a P_a(R, \theta_C, \theta_{\rm L})$. The requirement that such forms can be lifted to Poincaré invariant solutions of complete descent equations in four dimensions leads analogously to $\partial_{(a}P_{b)}(R, \theta_C, \theta_{\rm L}) = 0$ and $\partial_{i_A}P_a(R, \theta_C, \theta_{\rm L}) = 0$ where $\partial_a = \partial/\partial R^a$ and $\partial_{i_A} = \partial/\partial C^{i_A}$ (otherwise the lift would be obstructed at form-degree 3 by $F^{i_A}\star H^a\partial_{i_A}P_a(R, \theta_C, \theta_{\rm L})$ or at form-degree 4 by $H^a\star H^b\partial_a P_b(R, \theta_C, \theta_{\rm L})$). Now, in contrast to the C^I, the R^a are commuting variables and thus the general solution to the first condition, which has the structure of Killing vector equations in a flat space with coordinates R^a, is $P_a(R, \theta_C, \theta_{\rm L}) = P^{(2)}_a(\theta_C, \theta_{\rm L}) + \hat{R}^b P^{(3)}_{[ab]}(\theta_C, \theta_{\rm L})$ while the second condition imposes in addition that $P^{(2)}_a$ and $P^{(3)}_{[ab]}$ do not depend on the abelian ghosts. Let us now separately show how this leads to the representatives $f^{(2)}$ and $f^{(3)}$.

$f^{(2)}$ derives from $\tilde{H}^a P^{(2)}_a(\theta)$. Completing the Yang-Mills-θ's to the $\hat{q}$'s yields $\tilde{H}^a P^{(2)}_a(\hat{q}, \theta_{\rm L})$ (as $P^{(2)}_a(\theta)$ does not depend on the abelian $\hat{C}$'s). Since the $\tilde{H}$'s, $\hat{q}$'s with $m(r) > 2$ and Lorentz-θ's are $s_{\rm ext}$-invariant, one has

$$s_{\rm ext}\Big[\tilde{H}^a P^{(2)}_a(\hat{q}, \theta_{\rm L})\Big] = -\tilde{H}^a \sum_{r:m(r)=2} \mathrm{tr}_r(\mathcal{F}^2)\frac{\partial P^{(2)}_a(\hat{q}, \theta_{\rm L})}{\partial \hat{q}_r} \qquad \text{(B.42)}$$

where we used that $s_{\rm ext}\hat q_r = {\rm tr}_r\,\mathcal{F}^2$ for $m(r)=2$. The functions $\tilde H^a\,{\rm tr}_r(\mathcal{F}^2)$ which occur on the right hand side of (B.42) are $s_{\rm ext}$-closed elements of $\mathfrak{T}_{\rm inv}$ with ghost number 5. According to lemma 11 they are thus $s_{\rm ext}$-exact in $\mathfrak{T}_{\rm inv}$, i.e., there are functions X^a_r such that

$$\tilde H^a\,{\rm tr}_r(\mathcal{F}^2) = s_{\rm ext}X^a_r\,,\qquad X^a_r\in\mathfrak{T}_{\rm inv}\,. \tag{B.43}$$

Such functions are given explicitly in the lemma. Of course, they are determined only up to $s_{\rm ext}$-cocycles in $\mathfrak{T}_{\rm inv}$. However, this arbitrariness is irrelevant because adding such $s_{\rm ext}$-cocycles to X^a_r results at most in adding a solution $f^{(7)}$ and an $s_{\rm ext}$-coboundary to $f^{(2)}$ (since X^a_r has ghost number 4, lemma 11 implies that it is determined up to an $s_{\rm ext}$-cocycle of the form $\mathcal{P}\Omega^a_{r,1}+\bar{\mathcal{P}}\bar\Omega^a_{r,2}+s_{\rm ext}h^a_r$ (with $h^a_r\in\mathfrak{T}_{\rm inv}$) which gives rise to a representative $f^{(7)}$ up to an $s_{\rm ext}$-coboundary, see item (7) below).[26] The explicit computation of functions X^a_r is sketched at the end of the proof. Using (B.43), the right hand side of (B.42) gives

$$\begin{aligned}
-\tilde H^a\sum_{r:m(r)=2}{\rm tr}_r(\mathcal{F}^2)\,\frac{\partial P^{(2)}_a(\hat q,\theta_{\rm L})}{\partial\hat q_r} &= -\sum_{r:m(r)=2}(s_{\rm ext}X^a_r)\,\frac{\partial P^{(2)}_a(\hat q,\theta_{\rm L})}{\partial\hat q_r}\\
&= -s_{\rm ext}\sum_{r:m(r)=2}X^a_r\,\frac{\partial P^{(2)}_a(\hat q,\theta_{\rm L})}{\partial\hat q_r}+\\
&\quad+\sum_{\substack{r:m(r)=2\\ r':m(r')=2}}X^a_r\,{\rm tr}_{r'}(\mathcal{F}^2)\,\frac{\partial^2 P^{(2)}_a(\hat q,\theta_{\rm L})}{\partial\hat q_{r'}\partial\hat q_r}\,.
\end{aligned}\tag{B.44}$$

Since both X^a_r and ${\rm tr}_{r'}(\mathcal{F}^2)$ contain only terms with $\hat c$-degrees ≥ 2, the last term in (B.44) has $\hat c$-degree 4. Combining (B.42) and (B.44) and using part (i) of lemma 12, one concludes that this term vanishes and that $f^{(2)}$ is $s_{\rm ext}$-closed:

$$s_{\rm ext}f^{(2)} = \sum_{\substack{r:m(r)=2\\ r':m(r')=2}}X^a_r\,{\rm tr}_{r'}(\mathcal{F}^2)\frac{\partial^2 P^{(2)}_a(\hat q,\theta_{\rm L})}{\partial\hat q_{r'}\partial\hat q_r}=0. \tag{B.45}$$

$f^{(3)}$ derives similarly from $\tilde H^a\hat R^b P^{(3)}_{[ab]}(\theta)$. Again, we complete the Yang-Mills-θ's in $P^{(3)}_{[ab]}$ to the $\hat q$'s and compute the $s_{\rm ext}$-variation of the resultant function:

$$s_{\rm ext}\Big[\tilde H^a\hat R^b P^{(3)}_{[ab]}(\hat q,\theta_{\rm L})\Big] = -\Bigg[\tilde H^{[a}\mathcal{H}^{b]}+\tilde H^{[a}\hat R^{b]}\sum_{r:m(r)=2}{\rm tr}_r(\mathcal{F}^2)\frac{\partial}{\partial\hat q_r}\Bigg]P^{(3)}_{[ab]}(\hat q,\theta_{\rm L})\,. \tag{B.46}$$

The function $\tilde H^{[a}\mathcal{H}^{b]}$ is an $s_{\rm ext}$-closed element of $\mathfrak{T}_{\rm inv}$ with ghost number 4. According to lemma 11 it could only be nontrivial in $\mathfrak{T}_{\rm inv}$ if it were equivalent to a function of the form $\mathcal{P}\Omega_1+\bar{\mathcal{P}}\bar\Omega_2$. However, this is not the case: $\tilde H^{[a}\mathcal{H}^{b]}$ is quadratic in the fields of the linear multiplets and has dimension 0 (using dimension assignments as in the

[26] For analogous reasons the arbitrariness in the X's occurring in other representatives does not matter.

proof of lemma 3); hence, in order to be equivalent to a function $\mathcal{P}\Omega_1 + \bar{\mathcal{P}}\bar{\Omega}_2$, Ω_1 or Ω_2 would have to be given by $\bar{\mathcal{D}}^2 B(\hat{T})$ where $B(\hat{T})$ would be quadratic in the fields of the linear multiplets and have dimension 2; hence $B(\hat{T})$ would be a bilinear function in the ϕ^a; however, owing to the antisymmetry of $\tilde{H}^{[a}\mathcal{H}^{b]}$ in the indices a and b, this bilinear function would have to be proportional to $\phi^{[a}\phi^{b]}$ which vanishes since the ϕ's commute. Hence, the functions $\tilde{H}^{[a}\mathcal{H}^{b]}$ are s_{ext}-exact in $\mathfrak{T}_{\mathrm{inv}}$, i.e., there are functions $X^{[ab]} \in \mathfrak{T}_{\mathrm{inv}}$ such that

$$\tilde{H}^{[a}\mathcal{H}^{b]} = s_{\mathrm{ext}} X^{[ab]}\,, \qquad X^{[ab]} \in \mathfrak{T}_{\mathrm{inv}}\,. \tag{B.47}$$

Such functions are given explicitly in the lemma. Using (B.47) and (B.43), the right hand side of (B.46) gives

$$\begin{aligned}
-\Bigg[\tilde{H}^{[a}\mathcal{H}^{b]} + \tilde{H}^{[a}\hat{R}^{b]} \sum_{r:m(r)=2} \mathrm{tr}_r(\mathcal{F}^2)\frac{\partial}{\partial \hat{q}_r}\Bigg] P^{(3)}_{[ab]}(\hat{q},\theta_{\mathrm{L}}) &= \\
= -\Bigg[(s_{\mathrm{ext}} X^{[ab]}) + \sum_{r:m(r)=2} (s_{\mathrm{ext}} X^{[a}_r)\hat{R}^{b]}\,\frac{\partial}{\partial \hat{q}_r}\Bigg] P^{(3)}_{[ab]}(\hat{q},\theta_{\mathrm{L}}) \\
= -s_{\mathrm{ext}}\Bigg\{\Bigg[X^{[ab]} + \sum_{r:m(r)=2} X^{[a}_r\hat{R}^{b]}\frac{\partial}{\partial \hat{q}_r}\Bigg] P^{(3)}_{[ab]}(\hat{q},\theta_{\mathrm{L}})\Bigg\} + Z^{(3)}\,,
\end{aligned} \tag{B.48}$$

where

$$\begin{aligned}
Z^{(3)} = \sum_{r:m(r)=2} \Big[-X^{[ab]}\,\mathrm{tr}_r(\mathcal{F}^2) + X^{[a}_r\mathcal{H}^{b]}\Big]\,\frac{\partial P^{(3)}_{[ab]}(\hat{q},\theta_{\mathrm{L}})}{\partial \hat{q}_r} + \\
+ \sum_{\substack{r:m(r)=2\\ r':m(r')=2}} X^{[a}_r\hat{R}^{b]}\,\mathrm{tr}_{r'}(\mathcal{F}^2)\frac{\partial^2 P^{(3)}_{[ab]}(\hat{q},\theta_{\mathrm{L}})}{\partial \hat{q}_{r'}\partial \hat{q}_r}\,.
\end{aligned} \tag{B.49}$$

It is easy to verify that $Z^{(3)}$ has $\hat{c}$-degree 4 (in particular the term with $\hat{c}$-degree 3 in $X^{[a}_r\mathcal{H}^{b]}$ vanishes as it is proportional to $\phi^{[a}\phi^{b]}$). Analogously to (B.45), we thus conclude from (B.46) and (B.48), using again part (i) of lemma 12:

$$s_{\mathrm{ext}} f^{(3)} = Z^{(3)} = 0\,. \tag{B.50}$$

(4) The representatives $f^{(4)}$ arise from functions $k^{\Gamma}_{i_{\mathrm{f}}}\tilde{F}^{i_{\mathrm{f}}}P_{\Gamma}(\theta,\hat{R})$ with complex numbers $k^{\Gamma}_{i_{\mathrm{f}}}$, i.e., they involve the representatives $\tilde{F}^{i_{\mathrm{f}}}$ of $H(s_{\mathrm{susy}},\mathfrak{T}_{\mathrm{inv}})$. These functions give rise to solutions of complete non-supersymmetric Poincaré invariant descent equations in four dimensions only if they are of the form $\tilde{F}^{i_{\mathrm{f}}}\partial P^{(4)}(\theta,\hat{R})/\partial\hat{C}^{i_{\mathrm{f}}}$ where $P^{(4)}(\theta,\hat{R})$ can depend on all θ's except for those abelian $\hat{C}$'s which are not contained in $\{\hat{C}^{i_{\mathrm{f}}}\}$, see section 13 of [7] and section 8 of [14] (and also the brief discussion in item (2,3) above). Proceeding as in item (2,3), one obtains

$$s_{\mathrm{ext}}\Bigg[\tilde{F}^{i_{\mathrm{f}}}\frac{\partial P^{(4)}(\hat{q},\hat{C}_{\mathrm{free}},\theta_{\mathrm{L}},\hat{R})}{\partial \hat{C}^{i_{\mathrm{f}}}}\Bigg] = \tilde{F}^{i_{\mathrm{f}}}\Bigg[\mathcal{F}^{j_{\mathrm{f}}}\frac{\partial}{\partial \hat{C}^{j_{\mathrm{f}}}} + \mathcal{H}^a\frac{\partial}{\partial \hat{R}^a} + \sum_{r:m(r)=2}\mathrm{tr}_r(\mathcal{F}^2)\frac{\partial}{\partial \hat{q}_r}\Bigg]\times$$

$$\times \frac{\partial P^{(4)}(\hat{q}, \hat{C}_{\rm free}, \theta_{\rm L}, \hat{R})}{\partial \hat{C}^{i_{\rm f}}} \,. \tag{B.51}$$

Notice that the first term on the right hand side actually contains only the antisymmetrized products $\tilde{F}^{[i_{\rm f}}\mathcal{F}^{j_{\rm f}]}$ because $\partial^2 P^{(4)}/\partial\hat{C}^{j_{\rm f}}\partial\hat{C}^{i_{\rm f}}$ is antisymmetric in $i_{\rm f}$ and $j_{\rm f}$ owing to the odd Grassmann parity of the $\hat{C}$'s. Using lemma 11 one concludes that the $s_{\rm ext}$-closed functions $\tilde{F}^{[i_{\rm f}}\mathcal{F}^{j_{\rm f}]}$, $\tilde{F}^{i_{\rm f}}\mathcal{H}^a$ and $\tilde{F}^{i_{\rm f}}\,{\rm tr}_r(\mathcal{F}^2)$ which occur in (B.51) are $s_{\rm ext}$-exact in $\mathfrak{T}_{\rm inv}$: $\tilde{F}^{i_{\rm f}}\mathcal{H}^a$ and $\tilde{F}^{i_{\rm f}}\,{\rm tr}_r(\mathcal{F}^2)$ have ghost number 5 and 6, respectively, and are therefore $s_{\rm ext}$-exact in $\mathfrak{T}_{\rm inv}$ by the result of lemma 11 for $g \geq 5$; the $s_{\rm ext}$-exactness of $\tilde{F}^{[i_{\rm f}}\mathcal{F}^{j_{\rm f}]}$ in $\mathfrak{T}_{\rm inv}$ is seen using arguments as in the text after eq. (B.46): $\tilde{F}^{[i_{\rm f}}\mathcal{F}^{j_{\rm f}]}$ has ghost number 4 and dimension 0 and is quadratic in the $\hat{T}$'s of the super-Yang-Mills multiplets which implies that it cannot be equivalent to a function $\mathcal{P}\Omega_1 + \bar{\mathcal{P}}\bar{\Omega}_2$ because Ω_1 or Ω_2 would have to be proportional to $\hat{\lambda}^{[i_{\rm f}}\hat{\lambda}^{j_{\rm f}]}$ which vanishes. Hence lemma 11 implies that there are functions $X^{[i_{\rm f}j_{\rm f}]}$, $X^{i_{\rm f}a}$ and $X_r^{i_{\rm f}}$ such that

$$\begin{aligned}
\tilde{F}^{[i_{\rm f}}\mathcal{F}^{j_{\rm f}]} &= -s_{\rm ext}X^{[i_{\rm f}j_{\rm f}]}\,, \qquad X^{[i_{\rm f}j_{\rm f}]} \in \mathfrak{T}_{\rm inv}\,,\\
\tilde{F}^{i_{\rm f}}\mathcal{H}^a &= -s_{\rm ext}X^{i_{\rm f}a}\,, \qquad X^{i_{\rm f}a} \in \mathfrak{T}_{\rm inv}\,,\\
\tilde{F}^{i_{\rm f}}\,{\rm tr}_r(\mathcal{F}^2) &= -s_{\rm ext}X_r^{i_{\rm f}}\,, \qquad X_r^{i_{\rm f}} \in \mathfrak{T}_{\rm inv}\,.
\end{aligned} \tag{B.52}$$

Such functions are explicitly given in the lemma. Using the same reasoning that led us to eqs. (B.45) and (B.50) one concludes from (B.51) and (B.52) by means of part (i) of lemma 12 that $f^{(4)}$ is $s_{\rm ext}$-closed:

$$s_{\rm ext} f^{(4)} = 0\,. \tag{B.53}$$

(5) The representatives $f^{(5)}$ arise from functions $k^\Gamma_{ab}\tilde{H}^a\tilde{H}^b P_\Gamma(\theta,\hat{R})$ with complex numbers k^Γ_{ab}, i.e., they involve the representatives $\tilde{H}^a\tilde{H}^b$ of $H(s_{\rm susy},\mathfrak{T}_{\rm inv})$. In order to give rise to Poincaré invariant solutions of the complete non-supersymmetric descent equations, these functions must not involve abelian $\hat{C}$'s because otherwise the lifts of the BRST invariant 2-forms $k^\Gamma_{ab}(\star H^a)(\star H^b)P_\Gamma(\theta_C, C_{\rm Abel},\theta_{\rm L},\hat{R})$ to Poincaré invariant solutions of the descent equations were obstructed by the 4-forms

$$k^\Gamma_{ab}(\star H^a)(\star H^b)F^{i_A}\frac{\partial P_\Gamma(\theta_C, C_{\rm Abel},\theta_{\rm L},\hat{R})}{\partial C^{i_A}}\,.$$

This leaves us in this case with functions $\tilde{H}^a\tilde{H}^b P^{(5)}_{[ab]}(\theta,\hat{R})$ where $P^{(5)}_{[ab]}(\theta,\hat{R})$ does not depend on abelian $\hat{C}$'s. The antisymmetry of $P^{(5)}_{[ab]}$ in a and b is due to the fact that the $\tilde{H}$'s anticommute since they are Grassmann odd. Substituting the $\hat{q}$'s for the corresponding θ's and computing the $s_{\rm ext}$-variation of the resultant function gives

$$s_{\rm ext}\left[\tilde{H}^a\tilde{H}^b P^{(5)}_{[ab]}(\hat{q},\theta_{\rm L},\hat{R})\right] = \tilde{H}^a\tilde{H}^b\left[\mathcal{H}^c\frac{\partial}{\partial\hat{R}^c} + \sum_{r:m(r)=2}{\rm tr}_r(\mathcal{F}^2)\frac{\partial}{\partial\hat{q}_r}\right]P^{(5)}_{[ab]}(\hat{q},\theta_{\rm L},\hat{R})\,. \tag{B.54}$$

The functions $\tilde{H}^a\tilde{H}^b\mathcal{H}^c$ and $\tilde{H}^a\tilde{H}^b\,\mathrm{tr}_r(\mathcal{F}^2)$ are $s_{\rm ext}$-closed elements of $\mathfrak{T}_{\rm inv}$ with ghost numbers 5 and 6, respectively. Using lemma 11 we conclude that they are $s_{\rm ext}$-exact in $\mathfrak{T}_{\rm inv}$. Hence there are functions $X^{[ab]c}$ and $X_r^{[ab]}$ such that

$$\tilde{H}^a\tilde{H}^b\mathcal{H}^c = -s_{\rm ext}X^{[ab]c}\,,\qquad \tilde{H}^a\tilde{H}^b\,\mathrm{tr}_r(\mathcal{F}^2) = -s_{\rm ext}X_r^{[ab]}\,,\qquad X^{[ab]c},X_r^{[ab]}\in\mathfrak{T}_{\rm inv}\,. \tag{B.55}$$

Such functions are explicitly given in the lemma. Using (B.55), we obtain:

$$\begin{aligned}
&\tilde{H}^a\tilde{H}^b\left[\mathcal{H}^c\frac{\partial}{\partial\hat{R}^c}+\sum_{r:m(r)=2}\mathrm{tr}_r(\mathcal{F}^2)\frac{\partial}{\partial\hat{q}_r}\right]P^{(5)}_{[ab]}(\hat{q},\theta_{\rm L},\hat{R}) =\\
&\quad= -\left[(s_{\rm ext}X^{[ab]c})\frac{\partial}{\partial\hat{R}^c}+\sum_{r:m(r)=2}(s_{\rm ext}X_r^{[ab]})\frac{\partial}{\partial\hat{q}_r}\right]P^{(5)}_{[ab]}(\hat{q},\theta_{\rm L},\hat{R})\\
&\quad= -s_{\rm ext}\left[X^{[ab]c}\frac{\partial P^{(5)}_{[ab]}(\hat{q},\theta_{\rm L},\hat{R})}{\partial\hat{R}^c}+\sum_{r:m(r)=2}X_r^{[ab]}\frac{\partial P^{(5)}_{[ab]}(\hat{q},\theta_{\rm L},\hat{R})}{\partial\hat{q}_r}\right]+Z^{(5)}\,,
\end{aligned}\tag{B.56}$$

where

$$Z^{(5)} = X^{[ab]c}s_{\rm ext}\frac{\partial P^{(5)}_{[ab]}(\hat{q},\theta_{\rm L},\hat{R})}{\partial\hat{R}^c}-\sum_{r:m(r)=2}X_r^{[ab]}s_{\rm ext}\frac{\partial P^{(5)}_{[ab]}(\hat{q},\theta_{\rm L},\hat{R})}{\partial\hat{q}_r}\,. \tag{B.57}$$

$Z^{(5)}$ contains only terms with $\hat{c}$-degrees 3 and 4 because $X^{[ab]c}$ and $X_r^{[ab]}$ contain only terms with $\hat{c}$-degrees ≥ 2 and the $s_{\rm ext}$-variations in (B.57) contain only terms with $\hat{c}$-degrees ≥ 1. Furthermore $Z^{(5)}$ is $s_{\rm ext}$-closed because it is $s_{\rm ext}$-exact as one sees from (B.56) and (B.54). According to part (ii) of lemma 12, $Z^{(5)}$ is thus the $s_{\rm ext}$-variation of a function with $\hat{c}$-degree 4 whose δ_--variation is equal to the part of $Z^{(5)}$ with $\hat{c}$-degree 3. Explicitly one obtains

$$Z^{(5)} = -s_{\rm ext}\left[X^{[ab](cd)}\frac{\partial^2 P^{(5)}_{[ab]}(\hat{q},\theta_{\rm L},\hat{R})}{\partial\hat{R}^d\partial\hat{R}^c}\right], \tag{B.58}$$

with $X^{[ab](cd)}$ as given in the lemma. (B.54), (B.56) and (B.58) show that $f^{(5)}$ is $s_{\rm ext}$-closed:

$$s_{\rm ext}f^{(5)} = 0\,. \tag{B.59}$$

(6) The representatives $f^{(6)}$ arise from functions $(\mathcal{O}R^\Gamma(\hat{T}))P_\Gamma(\theta,\hat{R})$, i.e., they involve representatives $\mathcal{O}R(\hat{T})$ of $H(s_{\rm susy},\mathfrak{T}_{\rm inv})$. Substituting the $\hat{q}$'s for the corresponding θ's and computing the $s_{\rm ext}$-variation of the resultant function gives

$$\begin{aligned}
&s_{\rm ext}\left[(\mathcal{O}R^\Gamma(\hat{T}))P_\Gamma(\hat{q},\hat{C}_{\rm Abel},\theta_{\rm L},\hat{R})\right] =\\
&\quad= -(\mathcal{O}R^\Gamma(\hat{T}))\left[\mathcal{F}^{i_A}\frac{\partial}{\partial\hat{C}^{i_A}}+\mathcal{H}^a\frac{\partial}{\partial\hat{R}^a}+\sum_{r:m(r)=2}\mathrm{tr}_r(\mathcal{F}^2)\frac{\partial}{\partial\hat{q}_r}\right]P_\Gamma(\hat{q},\hat{C}_{\rm Abel},\theta_{\rm L},\hat{R})\,.
\end{aligned}\tag{B.60}$$

$(\mathcal{O}R^\Gamma(\hat{T}))\mathcal{F}^{i_A}$, $(\mathcal{O}R^\Gamma(\hat{T}))\mathcal{H}^a$ and $(\mathcal{O}R^\Gamma(\hat{T}))\,\mathrm{tr}_r(\mathcal{F}^2)$ are $s_{\rm ext}$-closed elements of $\mathfrak{T}_{\rm inv}$ with ghost numbers 5, 6 and 7, respectively. According to lemma 11 they are thus $s_{\rm ext}$-exact in $\mathfrak{T}_{\rm inv}$, i.e. there are functions $X^{\Gamma i_A}$, $X^{\Gamma a}$ and X^Γ_r such that

$$\begin{aligned}(\mathcal{O}R^\Gamma(\hat{T}))\mathcal{F}^{i_A} &= s_{\rm ext}X^{\Gamma i_A}\,, \qquad X^{\Gamma i_A}\in\mathfrak{T}_{\rm inv}\,,\\ (\mathcal{O}R^\Gamma(\hat{T}))\mathcal{H}^a &= s_{\rm ext}X^{\Gamma a}\,, \qquad X^{\Gamma a}\in\mathfrak{T}_{\rm inv}\,,\\ (\mathcal{O}R^\Gamma(\hat{T}))\,\mathrm{tr}_r(\mathcal{F}^2) &= s_{\rm ext}X^\Gamma_r\,, \qquad X^\Gamma_r\in\mathfrak{T}_{\rm inv}\,.\end{aligned} \tag{B.61}$$

Such functions are explicitly given in the lemma. Using the same reasoning that led us to eqs. (B.45) and (B.50) one concludes from (B.60) and (B.61) by means of part (i) of lemma 12 that $f^{(6)}$ is $s_{\rm ext}$-closed:

$$s_{\rm ext}f^{(6)}=0\,. \tag{B.62}$$

(7) The representatives $f^{(7)}$ arise from functions $(\mathcal{P}\Omega_1^\Gamma+\bar{\mathcal{P}}\bar{\Omega}_2^\Gamma)P_\Gamma(\theta,\hat{R})$, i.e., they involve representatives $(\mathcal{P}\Omega_1+\bar{\mathcal{P}}\bar{\Omega}_2)$ of $H(s_{\rm susy},\mathfrak{T}_{\rm inv})$. Proceeding as in item (6) we obtain

$$\begin{aligned}&s_{\rm ext}\Big[(\mathcal{P}\Omega_1^\Gamma+\bar{\mathcal{P}}\bar{\Omega}_2^\Gamma)\,P_\Gamma(\hat{q},\hat{C}_{\rm Abel},\theta_{\rm L},\hat{R})\Big]= \qquad\qquad (\mathrm{B.63})\\ &\quad=(\mathcal{P}\Omega_1^\Gamma+\bar{\mathcal{P}}\bar{\Omega}_2^\Gamma)\left[\mathcal{F}^{i_A}\frac{\partial}{\partial\hat{C}^{i_A}}+\mathcal{H}^a\frac{\partial}{\partial\hat{R}^a}+\sum_{r:m(r)=2}\mathrm{tr}_r(\mathcal{F}^2)\frac{\partial}{\partial\hat{q}_r}\right]P_\Gamma(\hat{q},\hat{C}_{\rm Abel},\theta_{\rm L},\hat{R})\,.\end{aligned}$$

Eq. (B.63) contains only terms with $\hat{c}$-degrees 3 and 4 because $\mathcal{P}\Omega_1^\Gamma+\bar{\mathcal{P}}\bar{\Omega}_2^\Gamma$, $\mathcal{F}^{i_A}$, $\mathcal{H}^a$ and $\mathrm{tr}_r(\mathcal{F}^2)$ contain only terms with $\hat{c}$-degrees ≥ 2, 1, 1 and 2, respectively. The terms with $\hat{c}$-degree 2 in $\mathcal{P}\Omega_1+\bar{\mathcal{P}}\bar{\Omega}_2$ are $4\mathrm{i}(\vartheta\vartheta\Omega_1-\bar{\vartheta}\bar{\vartheta}\bar{\Omega}_2)$, the terms with $\hat{c}$-degree 1 in $\mathcal{F}^{i_A}$ and $\mathcal{H}^a$ are $-\mathrm{i}(\vartheta\hat{\lambda}+\bar{\vartheta}\hat{\bar{\lambda}})^{i_A}$ and $-2\Theta\phi^a$, respectively. Owing to $\Theta=\vartheta\xi=\bar{\xi}\bar{\vartheta}$ and $\vartheta^\alpha\vartheta^\beta\vartheta^\gamma=\bar{\vartheta}^{\dot{\alpha}}\bar{\vartheta}^{\dot{\beta}}\bar{\vartheta}^{\dot{\gamma}}=0$ (the latter holds since the ϑ^α and $\bar{\vartheta}^{\dot{\alpha}}$ anticommute) the terms with $\hat{c}$-degree 3 in (B.63) are

$$f_3=4\left(\vartheta\vartheta\Omega_1^\Gamma\bar{\vartheta}\hat{\bar{\lambda}}^{i_A}-\bar{\vartheta}\bar{\vartheta}\bar{\Omega}_2^\Gamma\vartheta\hat{\lambda}^{i_A}\right)\frac{\partial P_\Gamma(\theta_C,R)}{\partial C^{i_A}}\,.$$

Part (ii) of lemma 12 implies thus that (B.63) is the $s_{\rm ext}$-variation of the function with $\hat{c}$-degree 4 whose δ_--variation equals f_3. Using (B.41) it is easy to identify this function:

$$\begin{aligned}f_3&=-\delta_-\left[8\Xi\left(\hat{\bar{\lambda}}^{i_A}\bar{\xi}\Omega_1^\Gamma+\xi\hat{\lambda}^{i_A}\bar{\Omega}_2^\Gamma\right)\frac{\partial P_\Gamma(\theta_C,R)}{\partial C^{i_A}}\right]\\ &=-\delta_-\left[8\Xi\left(\hat{\bar{\lambda}}^{i_A}\bar{\xi}\Omega_1^\Gamma+\xi\hat{\lambda}^{i_A}\bar{\Omega}_2^\Gamma\right)\frac{\partial P_\Gamma(\hat{q},\hat{C}_{\rm Abel},\theta_{\rm L},\hat{R})}{\partial\hat{C}^{i_A}}\right].\end{aligned} \tag{B.64}$$

Hence we conclude from (B.63), (B.64) and part (ii) of lemma 12 that $f^{(7)}$ is $s_{\rm ext}$-closed:

$$s_{\rm ext}f^{(7)}=0\,. \tag{B.65}$$

(8) The cocycles which arise from functions $k^{\Gamma}_{i_A}\mathcal{F}^{i_A}P_{\Gamma}(\theta,\hat{R})$ containing the representatives $\mathcal{F}^{i_A}$ of $H(s_{\rm susy},\mathfrak{T}_{\rm inv})$ are equivalent to linear combinations of representatives $f^{(6)}$ and $f^{(7)}$, as will be now shown. The lift of the BRST-invariant 2-form $F^{i_A}P_{i_A}(\theta_C,\theta_{\rm L},R)$ to a Poincaré invariant solution of complete non-supersymmetric descent equations in four dimensions is obstructed by the 4-form $F^{i_A}F^{j_A}\partial_{j_A}P_{i_A}(\theta_C,\theta_{\rm L},R)$ where $\partial_{i_A}=\partial/\partial C^{i_A}$. This obstruction is absent only if $\partial_{(j_A}P_{i_A)}(\theta_C,\theta_{\rm L},R)=0$. The general solution to this condition is $P_{i_A}(\theta_C,\theta_{\rm L},R)=\partial_{i_A}P(\theta_C,\theta_{\rm L},R)$, see section 13.2.2 of [7]. Hence functions $k^{\Gamma}_{i_A}\mathcal{F}^{i_A}P_{\Gamma}(\theta,\hat{R})$ can give rise to $s_{\rm ext}$-cocycles only if they are of the form $\mathcal{F}^{i_A}\partial P(\theta,\hat{R})/\partial\hat{C}^{i_A}$. Proceeding as in the other cases, one obtains

$$s_{\rm ext}\left[\mathcal{F}^{i_A}\frac{\partial P(\hat{q},\hat{C}_{\rm Abel},\theta_{\rm L},\hat{R})}{\partial\hat{C}^{i_A}}\right]=\mathcal{F}^{i_A}\left[\mathcal{H}^a\frac{\partial}{\partial\hat{R}^a}+\sum_{r:m(r)=2}{\rm tr}_r(\mathcal{F}^2)\frac{\partial}{\partial\hat{q}_r}\right]\times$$
$$\times\frac{\partial P(\hat{q},\hat{C}_{\rm Abel},\theta_{\rm L},\hat{R})}{\partial\hat{C}^{i_A}}\,. \tag{B.66}$$

The functions $\mathcal{F}^{i_A}\mathcal{H}^a$ and $\mathcal{F}^{i_A}\,{\rm tr}_r(\mathcal{F}^2)$ are $s_{\rm ext}$-coycles in $\mathfrak{T}_{\rm inv}$ with ghost numbers 5 and 6, respectively, and thus $s_{\rm ext}$-exact in $\mathfrak{T}_{\rm inv}$ according to lemma 11. Hence there are functions X^{i_Aa} and $X^{i_A}_r$ such that

$$\mathcal{F}^{i_A}\mathcal{H}^a=-s_{\rm ext}X^{i_Aa}\,,\qquad\mathcal{F}^{i_A}\,{\rm tr}_r(\mathcal{F}^2)=-s_{\rm ext}X^{i_A}_r\,,\qquad X^{i_Aa},X^{i_A}_r\in\mathfrak{T}_{\rm inv}\,. \tag{B.67}$$

Owing to $\mathcal{H}^a=-(1/2)\mathcal{O}\phi^a$, the first equation in (B.67) is just a special case of the first equation in (B.61). X^{i_Aa} can thus be obtained from $X^{\Gamma i_A}$ by choosing $R^{\Gamma}(\mathcal{T})\equiv\phi^a/2$. The computation of $X^{i_A}_r$ is very similar to the computation of the functions $X^{i_{\rm f}}_r$ which satisfy the last equation in (B.52). One obtains

$$X^{i_Aa}=-{\rm i}\,\Xi_\mu\,\hat{\lambda}^{i_A}\sigma^\mu\bar{\xi}\,\phi^a+\Xi\left(\frac{1}{4}\phi^a\mathcal{D}\hat{\lambda}^{i_A}+\hat{\lambda}^{i_A}\hat{\psi}^a\right)-{\rm c.c.}\,,$$
$$X^{i_A}_r=-{\rm i}\Xi\left[\bar{\hat{\lambda}}^{i_A}\bar{\xi}\,{\rm tr}_r(\hat{\lambda}\hat{\lambda})+2\hat{\lambda}^{i_A\alpha}\,{\rm tr}_r(\hat{\lambda}_\alpha\bar{\hat{\lambda}}\bar{\xi})\right]+{\rm c.c.} \tag{B.68}$$

Proceeding as in items (2,3) and (4), one obtains $s_{\rm ext}$-cocycles $f^{(8)}$:

$$s_{\rm ext}f^{(8)}=0\,,\qquad f^{(8)}=\left[\mathcal{F}^{i_A}+X^{i_Aa}\frac{\partial}{\partial\hat{R}^a}+\sum_{r:m(r)=2}X^{i_A}_r\frac{\partial}{\partial\hat{q}_r}\right]\frac{\partial P(\hat{q},\hat{C}_{\rm Abel},\theta_{\rm L},\hat{R})}{\partial\hat{C}^{i_A}}\,. \tag{B.69}$$

To show that $f^{(8)}$ is equivalent to a linear combination of representatives $f^{(6)}$ and $f^{(7)}$, as we have asserted above, we shall use the following relation:

$${\rm tr}_r(\mathcal{F}^2)=-\frac{\rm i}{8}\mathcal{P}\,{\rm tr}_r(\hat{\lambda}\hat{\lambda})+\frac{\rm i}{8}\bar{\mathcal{P}}\,{\rm tr}_r(\bar{\hat{\lambda}}\bar{\hat{\lambda}})+s_{\rm ext}\left[\Xi_\mu\,{\rm tr}_r(\hat{\lambda}\sigma^\mu\bar{\hat{\lambda}})\right]. \tag{B.70}$$

Furthermore we recall that one has

$$s_{\rm ext}P(\hat{q},\hat{C}_{\rm Abel},\theta_{\rm L},\hat{R})=\left[\mathcal{F}^{i_A}\frac{\partial}{\partial\hat{C}^{i_A}}+\mathcal{H}^a\frac{\partial}{\partial\hat{R}^a}+\sum_{r:m(r)=2}{\rm tr}_r(\mathcal{F}^2)\frac{\partial}{\partial\hat{q}_r}\right]\times$$
$$\times P(\hat{q},\hat{C}_{\rm Abel},\theta_{\rm L},\hat{R})\,. \tag{B.71}$$

Eqs. (B.69), (B.70) and (B.71) give (one may use part (i) of lemma 12 to verify this):

$$f^{(8)} - s_{\text{ext}} \left[P(\hat{q}, \hat{C}_{\text{Abel}}, \theta_{\text{L}}, \hat{R}) - \sum_{r:m(r)=2} \Xi_\mu \, \text{tr}_r(\hat{\lambda}\sigma^\mu \bar{\hat{\lambda}}) \frac{\partial P(\hat{q}, \hat{C}_{\text{Abel}}, \theta_{\text{L}}, \hat{R})}{\partial \hat{q}_r} \right] = \tag{B.72}$$

$$= \left[-\mathcal{H}^a + X^{i_A a} \frac{\partial}{\partial \hat{C}^{i_A}} + \sum_{r:m(r)=2} Y_r^a \frac{\partial}{\partial \hat{q}_r} + Y^{ab} \frac{\partial}{\partial \hat{R}^b} \right] \frac{\partial P(\hat{q}, \hat{C}_{\text{Abel}}, \theta_{\text{L}}, \hat{R})}{\partial \hat{R}^a} +$$

$$+ \sum_{r:m(r)=2} \left[\left\{ \frac{\mathrm{i}}{8} \mathcal{P} \, \text{tr}_r(\hat{\lambda}\hat{\lambda}) - \frac{\mathrm{i}}{8} \bar{\mathcal{P}} \, \text{tr}_r(\bar{\hat{\lambda}}\bar{\hat{\lambda}}) \right\} + Y_r^{i_A} \frac{\partial}{\partial \hat{C}^{i_A}} \right] \frac{\partial P(\hat{q}, \hat{C}_{\text{Abel}}, \theta_{\text{L}}, \hat{R})}{\partial \hat{q}_r},$$

where

$$Y_r^a = 4\Xi\phi^a \, \text{tr}_r(\xi\lambda\bar{\hat{\lambda}}\bar{\xi}), \qquad Y^{ab} = \mathrm{i}\Xi\phi^b(\xi\hat{\psi}^a - \bar{\hat{\psi}}^a\bar{\xi}),$$
$$Y_r^{i_A} = \mathrm{i}\Xi \left\{ \bar{\hat{\lambda}}^{i_A}\bar{\xi} \, \text{tr}_r(\hat{\lambda}\hat{\lambda}) - \xi\hat{\lambda}^{i_A} \, \text{tr}_r(\bar{\hat{\lambda}}\bar{\hat{\lambda}}) \right\}.$$

Recall that $\{P_\Gamma(\theta, \hat{R})\}$ denotes a basis for the monomials in the θ's and $\hat{R}$'s. Hence one has

$$\frac{\partial P(\theta, \hat{R})}{\partial \hat{R}^a} = k_a^\Gamma P_\Gamma(\theta, \hat{R}), \qquad \frac{\partial P(\theta, \hat{R})}{\partial \theta_r} = k^{r\Gamma} P_\Gamma(\theta, \hat{R})$$

for some complex numbers k_a^Γ and $k^{r\Gamma}$. (B.72) shows explicitly that $f^{(8)}$ is indeed equivalent to a linear combination of representatives $f^{(6)}$ and $f^{(7)}$ arising from the particular choices

$$R^\Gamma(\hat{T}) = \frac{1}{2} k_a^\Gamma \phi^a, \qquad \Omega_1^\Gamma = \sum_{r:m(r)=2} \frac{\mathrm{i}}{8} k^{r\Gamma} \, \text{tr}_r(\hat{\lambda}\hat{\lambda}), \qquad \bar{\Omega}_2^\Gamma = -\sum_{r:m(r)=2} \frac{\mathrm{i}}{8} k^{r\Gamma} \, \text{tr}_r(\bar{\hat{\lambda}}\bar{\hat{\lambda}}).$$

(9) To illustrate the derivation of the X-functions, let me finally describe in some detail how one derives X_r^a satisfying (B.43). (6.9) and (B.35) yield

$$\tilde{H}^a \, \text{tr}_r(\mathcal{F}^2) = \left(\mathrm{i}\bar{\hat{\psi}}\bar{\xi} - \mathrm{i}\xi\hat{\psi} + \hat{c}^\mu \hat{H}_\mu \right)^a \text{tr}_r \left(\hat{F} - \mathrm{i}\vartheta\hat{\lambda} - \mathrm{i}\bar{\vartheta}\bar{\hat{\lambda}} \right)^2. \tag{B.73}$$

We start from the terms with lowest $\hat{c}$-degree in this expression. They have $\hat{c}$-degree 2 and are given by

$$-\mathrm{i}\left(\bar{\hat{\psi}}\bar{\xi} - \xi\hat{\psi}\right)^a \text{tr}_r\left(\vartheta\hat{\lambda} + \bar{\vartheta}\bar{\hat{\lambda}}\right)^2 = -\mathrm{i}\left(\bar{\hat{\psi}}\bar{\xi} - \xi\hat{\psi}\right)^a \left[-\frac{1}{2}\vartheta\vartheta \, \text{tr}_r\left(\hat{\lambda}\hat{\lambda}\right) + 2\vartheta^\alpha\bar{\vartheta}^{\dot{\alpha}} \, \text{tr}_r\left(\hat{\lambda}_\alpha\bar{\hat{\lambda}}_{\dot{\alpha}}\right) - \right.$$
$$\left. - \frac{1}{2}\bar{\vartheta}\bar{\vartheta} \, \text{tr}_r\left(\bar{\hat{\lambda}}\bar{\hat{\lambda}}\right) \right]. \tag{B.74}$$

Eq. (B.74) is δ_--closed since it is the term of lowest $\hat{c}$-degree of an s_{ext}-cocycle. The δ_--cohomology (B.31) shows that it contains precisely two terms which are not δ_--exact. These are $(\mathrm{i}/2)\bar{\hat{\psi}}^a\bar{\xi}\vartheta\vartheta \, \text{tr}_r(\hat{\lambda}\hat{\lambda})$ and $(-\mathrm{i}/2)\xi\hat{\psi}^a\bar{\vartheta}\bar{\vartheta} \, \text{tr}_r(\bar{\hat{\lambda}}\bar{\hat{\lambda}})$ which are easily seen to be $\bar{b}$-exact and b-exact, respectively (owing to $b\phi = \xi\hat{\psi}$, $\bar{b}\phi = \bar{\hat{\psi}}\bar{\xi}$, $b\hat{\lambda} = \bar{b}\hat{\lambda} = 0$). Using that one has $s_{\text{ext}} f = (\delta_- + b + \bar{b} + \hat{c}^\mu \hat{\nabla}_\mu) f$ for $f \in \mathfrak{T}_{\text{inv}}$, we can thus write these terms as

$$\frac{\mathrm{i}}{2}\bar{\hat{\psi}}^a\bar{\xi}\vartheta\vartheta \, \text{tr}_r(\hat{\lambda}\hat{\lambda}) = (s_{\text{ext}} - b - \hat{c}^\mu\hat{\nabla}_\mu)\left[\frac{\mathrm{i}}{2}\vartheta\vartheta\phi^a \, \text{tr}_r(\hat{\lambda}\hat{\lambda})\right],$$

$$-\frac{\mathrm{i}}{2}\xi\hat{\psi}^a\bar{\vartheta}\bar{\vartheta}\,\mathrm{tr}_r(\bar{\hat{\lambda}}\bar{\hat{\lambda}}) = (s_{\mathrm{ext}} - \bar{b} - \hat{c}^\mu\hat{\nabla}_\mu)\left[-\frac{\mathrm{i}}{2}\bar{\vartheta}\bar{\vartheta}\phi^a\,\mathrm{tr}_r(\bar{\hat{\lambda}}\bar{\hat{\lambda}})\right]. \tag{B.75}$$

Using (B.75) in (B.73), one obtains

$$\tilde{H}^a\,\mathrm{tr}_r(\mathcal{F}^2) - s_{\mathrm{ext}}X^a_{r,2} = f^a_{r,2} + f^a_{r,3} + f^a_{r,4} \tag{B.76}$$

$$X^a_{r,2} = \frac{\mathrm{i}}{2}\vartheta\vartheta\phi^a\,\mathrm{tr}_r(\hat{\lambda}\hat{\lambda}) - \frac{\mathrm{i}}{2}\bar{\vartheta}\bar{\vartheta}\phi^a\,\mathrm{tr}_r(\bar{\hat{\lambda}}\bar{\hat{\lambda}})\,, \tag{B.77}$$

where $f^a_{r,2}$, $f^a_{r,3}$ and $f^a_{r,4}$ have $\hat{c}$-degree 2, 3 and 4, respectively, and $f^a_{r,2}$ is δ_--exact, i.e., there is some $X^a_{r,3}$ such that

$$\delta_- X^a_{r,3} = f^a_{r,2}\,. \tag{B.78}$$

(Of course, $X^a_{r,3}$ is only defined up to the δ_--variation of some function with $\hat{c}$-degree 4.) Explicitly one obtains:

$$\begin{aligned}
f^a_{r,2} &= \mathrm{i}\xi\hat{\psi}^a\left[-\frac{1}{2}\vartheta\vartheta\,\mathrm{tr}_r(\hat{\lambda}\hat{\lambda}) + 2\vartheta^\alpha\bar{\vartheta}^{\dot{\alpha}}\,\mathrm{tr}_r\left(\hat{\lambda}_\alpha\bar{\hat{\lambda}}_{\dot{\alpha}}\right)\right] - \frac{\mathrm{i}}{2}\vartheta\vartheta b\left[\phi^a\,\mathrm{tr}_r(\hat{\lambda}\hat{\lambda})\right] + \text{c.c.}\\
&= \mathrm{i}\xi\hat{\psi}^a\left[-\vartheta\vartheta\,\mathrm{tr}_r(\hat{\lambda}\hat{\lambda}) + 2\vartheta^\alpha\bar{\vartheta}^{\dot{\alpha}}\,\mathrm{tr}_r\left(\hat{\lambda}_\alpha\bar{\hat{\lambda}}_{\dot{\alpha}}\right)\right] - \frac{\mathrm{i}}{2}\vartheta\vartheta\phi^a\xi\mathcal{D}\,\mathrm{tr}_r(\hat{\lambda}\hat{\lambda}) + \text{c.c.}\\
X^a_{r,3} &= -\mathrm{i}\Xi_\mu\left(\bar{\xi}\bar{\sigma}^\mu\hat{\psi}^a + \frac{1}{2}\phi^a\bar{\xi}\bar{\sigma}^\mu\mathcal{D}\right)\mathrm{tr}_r(\hat{\lambda}\hat{\lambda}) - \mathrm{i}\Xi_\mu\xi\hat{\psi}^a\,\mathrm{tr}_r(\hat{\lambda}\sigma^\mu\bar{\hat{\lambda}}) + \text{c.c.}
\end{aligned} \tag{B.79}$$

Using (B.78) in (B.76), one obtains

$$\tilde{H}^a\,\mathrm{tr}_r(\mathcal{F}^2) - s_{\mathrm{ext}}(X^a_{r,2} + X^a_{r,3}) = \tilde{f}^a_{r,3} + \tilde{f}^a_{r,4}\,, \tag{B.80}$$

where $\tilde{f}^a_{r,3} = f^a_{r,3} - (b+\bar{b})X^a_{r,3}$ and $\tilde{f}^a_{r,4} = f^a_{r,4} - \hat{c}^\mu\hat{\nabla}_\mu X^a_{r,3}$. (B.80) is an s_{ext}-closed sum of terms with $\hat{c}$-degrees 3 and 4. According to part (ii) of lemma 12 it is thus the s_{ext}-variation of a function $X^a_{r,4}$ with $\hat{c}$-degree 4 that fulfills $\delta_- X^a_{r,4} = \tilde{f}^a_{r,3}$. We conclude that

$$\tilde{H}^a\,\mathrm{tr}_r(\mathcal{F}^2) - s_{\mathrm{ext}}(X^a_{r,2} + X^a_{r,3}) = s_{\mathrm{ext}}X^a_{r,4}\,, \tag{B.81}$$

where $\delta_- X^a_{r,4} = \tilde{f}^a_{r,3}$. The explicit computation gives

$$X^a_{r,4} = \Xi\left(\frac{1}{2}\hat{\psi}^a\mathcal{D} + \frac{1}{8}\phi^a\mathcal{D}^2\right)\mathrm{tr}_r(\hat{\lambda}\hat{\lambda}) - \frac{1}{2}\Xi\hat{H}^a_\mu\,\mathrm{tr}_r\left(\hat{\lambda}\sigma^\mu\bar{\hat{\lambda}}\right) + \text{c.c.} \tag{B.82}$$

Eq. (B.81) yields $X^a_r = X^a_{r,2} + X^a_{r,3} + X^a_{r,4}$. □

References

[1] P.L. White, *An analysis of the cohomology structure of super Yang-Mills coupled to matter*, *Class. and Quant. Grav.* **9** (1992) 1663.

[2] N. Maggiore, O. Piguet and S. Wolf, *Algebraic renormalization of $N = 1$ supersymmetric gauge theories*, *Nucl. Phys.* **B 458** (1996) 403 [hep-th/9507045].

[3] J. Gomis and S. Weinberg, *Are nonrenormalizable gauge theories renormalizable?*, *Nucl. Phys.* **B 469** (1996) 473 [hep-th/9510087].

[4] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications*, Cambridge University Press, Cambridge 1996.

[5] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in the antifield formalism, 1. General theorems*, *Commun. Math. Phys.* **174** (1995) 57 [hep-th/9405109].

[6] G. Barnich and M. Henneaux, *Consistent couplings between fields with a gauge freedom and deformations of the master equation*, *Phys. Lett.* **B 311** (1993) 123 [hep-th/9304057].

[7] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in gauge theories*, *Phys. Rept.* **338** (2000) 439 [hep-th/0002245].

[8] F. Brandt, N. Dragon and M. Kreuzer, *The gravitational anomalies*, *Nucl. Phys.* **B 340** (1990) 187.

[9] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in Einstein Yang-Mills theory*, *Nucl. Phys.* **B 455** (1995) 357 [hep-th/9505173].

[10] F. Brandt, *Local BRST cohomology and covariance*, *Commun. Math. Phys.* **190** (1997) 459 [hep-th/9604025].

[11] F. Brandt, *Jet coordinates for local BRST cohomology*, *Lett. Math. Phys.* **55** (2001) 149 [math-ph/0103006].

[12] F. Brandt, *Local BRST cohomology in minimal $D = 4$, $N = 1$ supergravity*, *Ann. Phys. (NY)* **259** (1997) 253 [hep-th/9609192].

[13] F. Brandt, M. Henneaux and A. Wilch, *Extended antifield formalism*, *Nucl. Phys.* **B 510** (1998) 640 [hep-th/9705007].

[14] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in the antifield formalism, II. Application to Yang-Mills theory*, *Commun. Math. Phys.* **174** (1995) 93 [hep-th/9405194].

[15] M. Henneaux, B. Knaepen and C. Schomblond, *Characteristic cohomology of p-form gauge theories*, *Commun. Math. Phys.* **186** (1997) 137 [hep-th/9606181].

[16] E. Noether, *Invariante Variationsprobleme*, *Nachrichten Kgl. Ges. d. Wiss. z. Göttingen, Math.-phys. Kl.* **2** (1918) 235.

[17] F. Brandt, *Anomaly candidates and invariants of $D = 4$, $N = 1$ supergravity theories*, *Class. and Quant. Grav.* **11** (1994) 849 [hep-th/9306054].

[18] O. Piguet and S.P. Sorella, *Algebraic renormalization: perturbative renormalization, symmetries and anomalies*, *Lect. Notes Phys.* **m28** (1995) 1.

[19] M. Henneaux, *Consistent interactions between gauge fields: the cohomological approach*, *Contemp. Math.* **219** (1998) 93 [hep-th/9712226].

[20] F. Brandt, *Deformations of global symmetries in the extended antifield formalism*, *J. Math. Phys.* **40** (1999) 1023 [hep-th/9804153].

[21] R. Haag, J.T. Lopuszanski and M. Sohnius, *All possible generators of supersymmetries of the S matrix*, *Nucl. Phys.* **B 88** (1975) 257.

[22] H. Nicolai and P.K. Townsend, *$N = 3$ supersymmetry multiplets with vanishing trace anomaly: building blocks of the $N > 3$ supergravities*, *Phys. Lett.* **B 98** (1981) 257.

[23] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, *Ten-dimensional Maxwell-Einstein supergravity, its currents and the issue of its auxiliary fields*, *Nucl. Phys.* **B 195** (1982) 97.

JHEP04(2003)035

[24] G.F. Chapline and N.S. Manton, *Unification of Yang-Mills theory and supergravity in ten-dimensions*, *Phys. Lett.* **B 120** (1983) 105.

[25] M.B. Green and J.H. Schwarz, *Anomaly cancellation in supersymmetric $D = 10$ gauge theory and superstring theory*, *Phys. Lett.* **B 149** (1984) 117.

[26] G. Barnich, M. Henneaux and R. Tatar, *Consistent interactions between gauge fields and the local BRST cohomology: the example of Yang-Mills models*, *Int. J. Mod. Phys.* **D 3** (1994) 139 [hep-th/9307155].

[27] C. Bizdadea and S.O. Saliu, *A note on 'irreducible' p-form gauge theories with Stueckelberg coupling*, *Europhys. Lett.* **41** (1998) 129 [hep-th/9911143].

[28] V.I. Ogievetsky and I.V. Polubarinov, *The notoph and its possible interactions*, *Sov. J. Nucl. Phys.* **4** (1967) 156.

[29] D.Z. Freedman and P.K. Townsend, *Antisymmetric tensor gauge theories and nonlinear sigma models*, *Nucl. Phys.* **B 177** (1981) 282.

[30] U. Lindström and M. Roček, *Scalar tensor duality and $N = 1$, $N = 2$ nonlinear sigma models*, *Nucl. Phys.* **B 222** (1983) 285.

[31] T.E. Clark, C.H. Lee and S.T. Love, *Supersymmetric tensor gauge theories*, *Mod. Phys. Lett.* **A 4** (1989) 1343.

[32] F. Brandt and U. Theis, *$D = 4$, $N = 1$ supersymmetric Henneaux-Knaepen models*, *Nucl. Phys.* **B 550** (1999) 495 [hep-th/9811180].

[33] P. Fayet and J. Iliopoulos, *Spontaneously broken supergauge symmetries and Goldstone spinors*, *Phys. Lett.* **B 51** (1974) 461.

[34] O. Piguet and K. Sibold, *The anomaly in the Slavnov identity for $N = 1$ supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 247** (1984) 484.

[35] T.E. Clark and S.T. Love, *Supersymmetric effective actions for anomalous internal chiral symmetries*, *Phys. Lett.* **B 138** (1984) 289.

[36] N.K. Nielsen, *Anomalies of supersymmetric chiral Yang-Mills currents*, *Nucl. Phys.* **B 244** (1984) 499.

[37] R. Garreis, M. Scholl and J. Wess, *On the supersymmetric, nonabelian chiral anomalies*, *Z. Physik* **C 28** (1985) 623.

[38] G. Girardi, R. Grimm and R. Stora, *Chiral anomalies in $N = 1$ supersymmetric Yang-Mills theories*, *Phys. Lett.* **B 156** (1985) 203.

[39] L. Bonora, P. Pasti and M. Tonin, *ABJ anomalies in supersymmetric Yang-Mills theories*, *Phys. Lett.* **B 156** (1985) 341.

[40] E. Guadagnini, K. Konishi and M. Mintchev, *Nonabelian chiral anomalies in supersymmetric gauge theories*, *Phys. Lett.* **B 157** (1985) 37.

[41] S. Ferrara, L. Girardello, O. Piguet and R. Stora, *The nonpolynomial structure of supersymmetric chiral anomalies*, *Phys. Lett.* **B 157** (1985) 179.

[42] L. Bonora, P. Pasti and M. Tonin, *The consistent chiral anomaly in supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 261** (1985) 249.

JHEP04(2003)035

[43] H. Itoyama, V.P. Nair and H.-c. Ren, *Supersymmetry anomalies and some aspects of renormalization*, *Nucl. Phys.* **B 262** (1985) 317.

[44] K. Harada and K. Shizuya, *Nonabelian consistent chiral anomalies in supersymmetric gauge theories*, *Phys. Lett.* **B 162** (1985) 322.

[45] O. Piguet and K. Sibold, *Renormalized supersymmetry. The perturbation theory of $N = 1$ supersymmetric theories in flat space-time*, Progress in Physics **12**, Birkhäuser, Boston 1986.

[46] M. Pernici and F. Riva, *Chiral anomalies in supersymmetric theories*, *Nucl. Phys.* **B 267** (1986) 61.

[47] I.N. McArthur and H. Osborn, *Supersymmetric chiral effective action and nonabelian anomalies*, *Nucl. Phys.* **B 268** (1986) 573.

[48] E. Guadagnini and M. Mintchev, *Chiral anomalies and supersymmetry*, *Nucl. Phys.* **B 269** (1986) 543.

[49] H. Itoyama, V.P. Nair and H.-c. Ren, *Supersymmetry anomalies: further results*, *Phys. Lett.* **B 168** (1986) 78.

[50] V.K. Krivoshchekov, P.B. Medvedev and L.O. Chekhov, *Explicit form of nonabelian consistent chiral SUSY anomaly*, *Theor. Math. Phys.* **68** (1987) 796.

[51] P. Altevogt and R. Kaiser, *Supersymmetric gauge anomalies in two-dimensions, four-dimensions, six-dimensions and ten-dimensions*, *Z. Physik* **C 43** (1989) 455.

[52] M. Marinkovic, *Wess-Zumino effective action for supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 366** (1991) 74.

[53] Y. Ohshima, K. Okuyama, H. Suzuki and H. Yasuta, *Remark on the consistent gauge anomaly in supersymmetric theories*, *Phys. Lett.* **B 457** (1999) 291 [hep-th/9904096].

[54] S.J. Gates Jr., M.T. Grisaru and S. Penati, *Holomorphy, minimal homotopy and the 4d, $N = 1$ supersymmetric Bardeen-Gross-Jackiw anomaly terms*, *Phys. Lett.* **B 481** (2000) 397 [hep-th/0002045].

[55] S.J. Gates Jr., M.T. Grisaru, M.E. Knutt, S. Penati and H. Suzuki, *Supersymmetric gauge anomaly with general homotopic paths*, *Nucl. Phys.* **B 596** (2001) 315 [hep-th/0009192].

[56] L.A. Dickey, *Soliton equations and Hamiltonian systems*, *Adv. Ser. Math. Phys.* **12** (1991) 1.

[57] G. Barnich and M. Henneaux, *Isomorphisms between the Batalin-Vilkovisky antibracket and the Poisson bracket*, *J. Math. Phys.* **37** (1996) 5273 [hep-th/9601124].

[58] T. Uematsu and C.K. Zachos, *Structure of phenomenological lagrangians for broken supersymmetry*, *Nucl. Phys.* **B 201** (1982) 250.

[59] S. Samuel and J. Wess, *A superfield formulation of the nonlinear realization of supersymmetry and its coupling to supergravity*, *Nucl. Phys.* **B 221** (1983) 153.

[60] T.E. Clark and S.T. Love, *Goldstino couplings to matter*, *Phys. Rev.* **D 54** (1996) 5723 [hep-ph/9608243].

[61] F. Brandt, *Lagrangians and anomaly candidates of $D = 4$, $N = 1$ rigid supersymmetry*, *Nucl. Phys.* **B 392** (1993) 428.

[62] M. Dubois-Violette, M. Henneaux, M. Talon and C.-M. Viallet, *General solution of the consistency equation*, *Phys. Lett.* **B 289** (1992) 361 [hep-th/9206106].

JHEP04(2003)035

[63] N. Dragon, *BRS symmetry and cohomology*, hep-th/9602163.

[64] J.A. Dixon and R. Minasian, *BRS cohomology of the supertranslations in $D = 4$*, *Commun. Math. Phys.* **172** (1995) 1 [hep-th/9304035].

[65] B. Zumino, *Chiral anomalies and differential geometry*, in *Relativity, Groups and Topology II*, B.S. DeWitt and R. Stora eds., North-Holland, Amsterdam 1984.